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A Characterization of Some Finite Groups of Characteristic 3

MICHAEL ASCHBACHER*

Department of Mathematics, California Institute of Technology, Pasadena, California 91125

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Denote by $\Omega_n^\varepsilon(3)$, $\varepsilon = \pm$, the nonabelian composition factor of the isometry group of a nondegenerate n-dimensional orthogonal space over $GF(3)$, with maximal possible Witt index when $\varepsilon = +$ or n is odd, and Witt index $(n/2) - 1$ when n is even and $\varepsilon = -$. $\Omega_6^-(3)/Z_2$ denotes the perfect central extension of $\Omega_6^-(3)$ over a center of order 2. In this paper we essentially solve the standard form problem for $\Omega_n^\varepsilon(3)$, $n \leqslant 5 \leqslant 8$, and $\Omega_6^-(3)/Z_2$. The actual theorem established is weaker, but seems to be the appropriate result for the purposes of interesting applications. With the exception of $\Omega_6^-(3)$, each of these standard form problems has already been solved; see, for example, Gomi [10] and Foote [7] for $\Omega_5(3)$, Suzuki [13] for $\Omega_6^+(3)$, and Walter [15] for $\Omega_n^\varepsilon(3)$, $n = 7, 8$, an $\Omega_6^-(3)/Z_2$. There are even partial results available for $\Omega_6^-(3)$ due to Finkelstein [6]. On the other hand the treatment here appears, in most cases, to be significantly shorter and more self-contained. In part this can be explained by the restricted hypothesis.

The motivation for the problem comes from the theory of groups of component type. The reader is directed to [1, 11] for a discussion of this theory and the definition of the basic terminology of the subject. There are two important applications of the Main Theorem to the theory of groups of component type. First, it completes the classification of the finite groups possessing an involution whose centralizer has a component whose nonabelian composition factor is a group of Lie type and odd characteristic. Second, it completes the verification of the B-conjecture in at least one, and possibly both, of the current approaches to that conjecture.

Let G be a finite group $\mathscr{I}(G)$ denotes the set of involutions of G and $\mathscr{C}(G)$ the set of components of G. $\mathscr{C}(\mathscr{I}(G))$ consists of the groups in $\mathscr{C}(C_G(x))$ as x varies over $\mathscr{I}(G)$.

Let $\mathscr{F}$ denote the family of quasisimple groups L with $L/O(L) \cong \Omega_n^\varepsilon(3)$,

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$5 \leqslant n \leqslant 8$, or $\Omega_6^-(3)/Z_2$. $\mathscr{F}^*$ consists of $\mathscr{F}$ together with the quasisimple groups L with $L/O(L) \cong L_2(9)$, $L_2(81)$, $L_3(3)$ or $U_3(3)$.

Consider the following hypothesis:

HYPOTHESIS A. *G is a finite group with $F^*(G)$ simple such that*

(1) *G satisfies the B-conjecture, and*

(2) *if $x, y \in \mathscr{I}(G)$ with $[x, y] = 1$, $L \in \mathscr{C}(C_G(\langle x, y \rangle))$, $K \in \mathscr{C}(C_G(y))$, $L \leqslant KK^x$, and $L \in \mathscr{F}^*$, then $K \in \mathscr{F}^*$.*

MAIN THEOREM. *Let G satisfy Hypothesis* A *and assume $\mathscr{C}(\mathscr{I}(G))$ contains a member in $\mathscr{F}$. Then $F^*(G) \in$ Chev(3) or $F^*(G) \cong L_4(4)$.*

Recall Chev(p) consists of the finite simple groups of Lie type over fields of characeristic p.

The definition of $\mathscr{F}^*$ is somewhat arbitrary. Following Harris in [11], we might define the closure $\mathrm{Cl}(\mathscr{E})$ of a family $\mathscr{E}$ of quasisimple groups to be the smallest family $\mathscr{E}'$ containing $\mathscr{E}$ such that whenever X is quasisimple and $\mathscr{C}(\mathscr{I}(\mathrm{Aut}(X)))$ contains a member of $\mathscr{E}'$, then $X \in \mathscr{E}'$. Throughout most of the proof it would probably suffice to know $\mathscr{F}^* \subseteq \mathrm{Cl}(\mathscr{F} \cup \{L_2(9), L_3(3), U_3(3)\})$, to know $\mathrm{Cl}(\mathscr{F}) \cap \mathscr{F}^* = \mathscr{F}$, and to know that if $L \in \mathscr{F}^*$ with $L/Z(L) \in$ Chev(3) then $L \in \mathscr{F}$ or $L/O(L) \cong L_2(3^n)$, $L_3(3)$, or $U_3(3)$. The family $\mathscr{F}' = \mathscr{F} \cup \{L_2(9), L_3(3), U_3(3)\}$ is significant as

$$\mathscr{F}' = \bigcup_{X \in \mathscr{F}'} \mathscr{C}(\mathscr{I}(\mathrm{Aut}(X)).$$

The justification for excluding most of Chev(3) will be discussed below. The hypothesis just mentioned seems to be sufficient to support the crucial Lemma 3.4. Lemmas 6.1, 6.3, and 6.4 require more restraints on $\mathscr{F}^*$; still much latitude remains, and in any event these lemmas are less crucial. On the other hand the definition of $\mathscr{F}^*$ given here is convenient and seems to suffice for the purposes of classifying the finite simple groups.

Let G be a finite group such that $\mathscr{C}(\mathscr{I}(G))$ possesses a member L with $L/Z(L) \in$ Chev(p), p odd. Unless $L \cong L_2(q)$ or ${}^2G_2(q)$, there is $z \in \mathscr{I}(L)$ and $K \trianglelefteq\trianglelefteq C_L(z)$ with $z \in K \cong SL_2(q)$. If $K \trianglelefteq\trianglelefteq C_G(z)$ then Corollary III in [3] shows $F^*(G) \in$ Chev(p). Hence the cases $L \cong L_2(q)$ or ${}^2G_2(q)$ require special treatment. In the remaining cases it is usually possible to show that L and z may be picked so that $K \trianglelefteq\trianglelefteq C_G(z)$. See, for example, Harris [11] and Walter [15]. However, when $q = 3$ the fact that $SL_2(3)$ is not quasisimple causes some problems. Hence the groups over $GF(3)$ require special attention. Even here, if the Lie rank of L is not too small, the goal is achieved with reasonable elegance; again see Harris or Walter. But when the Lie rank is small and $L/O(L) \cong \Omega_n^\varepsilon(3)$, $n \leqslant 8$, $\Omega_6^-(3)/Z_2$, $L_3(3)$, $U_3(3)$, or $G_2(3)$, the methods of Harris and Walter begin to break down.

The basic approach in this paper is to use other methods to show $K \trianglelefteq\trianglelefteq C_G(z)$. It is presumably possible to handle $L_3(3)$, $U_3(3)$, and $G_2(3)$ via this approach also, but because of the small size of these groups the techniques are not so effective, and I suspect in the end one would be left with a treatment very similar to the existing treatments.

Another important technique is to push up maximal elementary abelian 2-subgroups. This approach is by now quite standard. What does not appear to be standard is the use of this technique in conjunction with Corollary III in [3] and an analysis of $\mathscr{C}(\mathscr{I}(G))$ from the point of view of the component ordering. The result is a reasonable short and simple treatment of the standard form problems for the small orthogonal groups.

It should be pointed out that the treatment here of $\Omega_8^+(3)$ is substantially the same as Walter's in [15] and was suggested by Walter's proof.

Some of the notation used here is not entirely standard. $L_n^+(q) = L_n^-(q)$ and $L_n^-(q) = U_n(q)$. Given a 2-group S, $\mathscr{A}(S)$ denotes the set of elementary abelian subgroups of S of maximal order and $J(S) = \langle \mathscr{A}(S) \rangle$. $A_1/A_2/\cdots$ denotes a group G with normal series $G = G_1 \trianglerighteq G_2 \trianglerighteq \cdots$ and $G_i/G_{i+1} \cong A_i$.

Finally we record two lemmas which will be needed.

(1.1) *Let T be a 2-group and X a subgroup of T with $\Phi(X) \neq 1$ which is a TI-set in T. Then $\langle X, X^t \rangle = X \times X^t$ for $t \in T - N(X)$.*

Proof. See 1.3 in [15].

(1.2) *Let T be a 2-group, $t \in \mathscr{I}(T)$, and $C_T(t) \cong Z_2 \times Z_4$. Then T is of sectional 2-rank at most 3.*

Proof. See 2.1 in [16].

2

In this section, V is an n-dimensional orthogonal space of Witt index ε over $GF(3)$ with quadratic form f and induced bilinear form

$$(x, y) = (f(x+y) - f(x) - f(y))/2.$$

We choose notation so that V has a basis $\mathscr{Y} = \{y_1, \ldots, y_n\}$ such that the matrix $((y_i, y_j))$ is

$$\begin{bmatrix} & I_m & \\ I_m & & \\ & & 1 \end{bmatrix} \quad \text{if} \quad n = 2m+1,$$

$$\begin{bmatrix} & I_m \\ I_m & \end{bmatrix} \quad \text{if} \quad n = 2m \quad \text{and} \quad \varepsilon = +,$$

$$\begin{bmatrix} & I_m & \\ I_m & & \\ & & I_2 \end{bmatrix} \quad \text{if} \quad n = 2m \quad \text{and} \quad \varepsilon = -,$$

where I_k is the $k \times k$ identity matrix.

Let $\Gamma = \Gamma O(V)$ be the group of all nonsingular linear transformations σ of V such that $f(v^\sigma) = \lambda_\sigma f(v)$ for each $v \in V$ and $\lambda_\sigma \in GF(3)^\#$ depending on σ but not v. Let $G = O(V) \leqslant \Gamma$ be the orthogonal group on V consisting of those σ in Γ with $\lambda_\sigma = 1$. Let $\Omega = \Omega(V) = E(G)$, $\langle \pi \rangle = Z(\Gamma)$ and $\Gamma^* = \Gamma / \langle \pi \rangle$. Then $\langle \pi \rangle \cong Z_2$ and $\pi \in \Omega$ precisely when $n \equiv 0 \bmod 4$ and $\varepsilon = +$ or $n \equiv 2 \bmod 4$ and $\varepsilon = -$. $\Omega^* \cong \Omega_n^\varepsilon(3)$ and $G^* \cong PO_n^\varepsilon(3)$. By 15.9 in [3],

(2.1) *Either* $\mathrm{Aut}(\Omega^*) = \Gamma^*$ *or* $n = 8$, $\varepsilon = +$ *and* $\mathrm{Aut}(\Omega^*) = \langle \Gamma^*, \xi \rangle$, *where* ξ *induces a graph automorphism on* Ω^*.

Moreover it is well known that

(2.2) (1) $\Gamma^* = G^*$ *and* $|G^* \colon \Omega^*| = 2$ *if* n *is odd.*

(2) $|\Gamma^* \colon G^*| = 2$ *if* n *is even.*

(3) $\Gamma^*/\Omega^* \cong E_4$ *if* $n \equiv 0 \bmod 4$ *and* $\varepsilon = -$ *or* $n \equiv 2 \bmod 4$ *and* $\varepsilon = +$.

(4) $\Gamma^*/\Omega^* \cong D_8$ *and* $G^*/\Omega^* \cong E_4$ *if* $n \equiv 0 \bmod 4$ *and* $\varepsilon = +$ *or* $n \equiv 2 \bmod 4$ *and* $\varepsilon = -$.

The involutions in Γ^* are discussed in [3, pp. 401, 402]. In particular if x^* is an involution in G^* which is the image of an involution x in G, then V is the orthogonal sum of the nondegenerate subspaces $C_V(x)$ and $[V, x]$, and the conjugacy class of x in G (or under Ω) is determined by the isometry type of these subspaces. Moreover the isometry type of a nondegenerate subspace U is determined by its sign and dimension. Here if $U = \langle v \rangle$ is of dimension one then $\mathrm{sgn}(\langle v \rangle) = f(v)$, while more generally we may write U as an orthogonal sum

$$U = \langle v_1 \rangle \oplus \cdots \oplus \langle v_r \rangle$$

of nondegenerate 1-dimensional subspaces and we define

$$\mathrm{sgn}(U) = \left(\prod_{i=1}^{r} f(v_i) \right) (-1)^{[r/2]}$$

In particular if $\dim(U)$ is even then $\mathrm{sgn}(U)$ is the sign of the Witt index of U. Denote by $i(r, \alpha)$ the set of involutions x^* of G^* such that x^* is the image of an involution x in G with $[V, x]$ of dimension r and sign α. If $\pi \in \Omega$ we also are interested in the involutions of Ω and we use the same convention for such involutions. Notice that

(2.3) *If x is an involution in G with $[V, x]$ of dimension r and* sign α *then $[V, \pi x]$ is of dimension $n-r$ and* sign α *if n is odd, $\alpha(-1)^r$ if n is even and $\varepsilon=+$, and $\alpha(-1)^{r+1}$ if n is even and $\varepsilon=-$.*

As a consequence we conclude

(2.4) *In G^* we have*

(1) *$i(r, \alpha)=i(n-r, \alpha)$ if n is odd.*

(2) *$i(r, \alpha)=i(n-r, (-1)^r \alpha)$ if n is even and $\varepsilon=+$.*

(3) *$i(r, \alpha)=i(n-r, (-1)^{r+1} \alpha)$ if n is even and $\varepsilon=-$.*

One can also check that

(2.5) *If r is odd and $\gamma \in \Gamma^*-G^*$ then $i(r, \alpha)^\gamma=i(r, -\alpha)$,*

(2.6) *Either*

(1) *$C_{G^*}(x^*)=(O([V, X]) \times O(C_V(x)))^*$, or*

(2) *n is even, $\varepsilon=+$, $r=n/2$ and*

$$C_{G^*}(x) \cong (O[V, X] \text{ wreath } Z_2)/Z(O([V, X] \text{ wreath } Z_2)).$$

By 15.12 in [3]

(2.7) (1) *Either n is even and $\varepsilon=(-1)^{n/2}$ or every involution in G^* is the image of an involution in G.*

(2) *If n is even and $\varepsilon=(-1)^{n/2}$ then G^* has a unique class $(p^*)^G$ of projective involutions. $C_{\Gamma^*}(p^*)$ is the split extension of $GL^{\varepsilon}_{n/2}(3)$ by a graph automorphism.*

Here p^* is projective if $p^2=\pi$.

Next from [3, p. 400], $\Gamma=\langle\rho, G\rangle$, where, with respect to the basis $\mathscr{Y}$,

$$\rho=\begin{bmatrix} -I_m & \\ & I_m \end{bmatrix} \qquad \text{if} \quad n=2m \quad \text{and} \quad \varepsilon=+,$$

$$\rho=\begin{bmatrix} -I_m & & \\ & I_m & \\ & & \rho_2 \end{bmatrix} \qquad \text{if} \quad n=2m \quad \text{and} \quad \varepsilon=-,$$

$$\rho_2=\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

We can now calculate to determine

(2.8) *If* $n=8$ *and* $\varepsilon=+$ *then every involution in* Γ^* *is conjugate under* $\mathrm{Aut}(\Omega^*)$ *to an involution in* $i(r, \pm 1)$, $1 \leqslant r \leqslant 4$.

Recall here that in this special case $\mathrm{Aut}(\Omega^*) \neq \Gamma^*$, so the graph automorphism of order 3 supplies extra fusion.

Next consider the case where $n=6$ and $\varepsilon=-$. Let z be the involution in Ω with $[V,z]=\langle y_i: 1 \leqslant i \leqslant 4\rangle$. An easy calculation shows $C_{G^*}(z^*)$ is transitive on the involutions in $\rho^* C_{G^*}(z^*)$, and if x^* is such an involution then $C_{\Omega^*}(\langle x^*, z^*\rangle) \cong SD_{16}$ and $|C_{G^*}(\langle x^*, z^*\rangle)|=32$, from which we conclude

(2.9) *If* $n=6$ *and* $\varepsilon=-$ *then there is one class* γ^{G^*} *of involutions in* Γ^*-G^* *and* $C_{G^*}(\gamma) \cong \mathrm{Aut}(A_6)$.

We can now list in Table 2.10 the Ω^*-conjugacy classes of involutions in $\Gamma^* \cong \mathrm{Aut}(\Omega_6^-(3))$, the centralizer in Γ^* of a representative, and the coset of Γ^*/Ω^* of the class.

We next consider the case $n=6$, $\varepsilon=+$. In this case we find $C_{\Omega^*}(z^*)$ has two orbits on the involutions in $\rho^* C_{G^*}(z^*)$ with representatives ρ^* and $\rho_0^*=\rho^* x^* y^*$, where x^* is of type $i(1,\alpha)$ in $C(O^2(C(z^*)))$ and y^* is of order 4 in $O^2(C(\langle z^*, \rho^*\rangle))$. By 15.21 in [3], $C_{G^*}(\rho^*)$ is $\mathrm{Aut}(L_3(3))$. $C_{\Omega^*}(\langle \rho_0^*, z^*\rangle) \cong D_{16}$ and $|C_{G^*}(\langle \rho_0^*, z^*\rangle)|=32$, so we conclude $C_{G^*}(\rho_0^*) \cong \mathrm{Aut}(A_6)$.

We can now list in Table 2.11 the Ω^*-conjugacy classes of involutions in $\Gamma^* \cong \mathrm{Aut}(\Omega_6^+(3))$, the centralizer in Γ^* of a representative, and the coset of Γ^*/Ω^* of the class.

In the remainder of this section continue to let z be the involution with $[V,z]=\langle y_i: 1 \leqslant i \leqslant 4\rangle$. Then z is of type $i(4,+)$ and $[V,z]$ is a nondegenerate orthogonal space of dimension 4 and sign $+$. The structure

TABLE 2.10

Involutions and centralizers in $\mathrm{Aut}(\Omega_6^-(3))$

Involution	Centralizer	Coset
$i(4,+)$	$Z_2/(D_8 \cdot (E_4/SL_2(3)^* SL_2(3)))$	Ω^*
$i(4,-)$	$Z_2/(E_4 \times S_6)$	$Z(\Gamma/\Omega)$
p^*	$Z_2/(Z_4 \times U_3(3))$	$Z(\Gamma/\Omega)$
$i(1,+)$	$Z_2 \times PO_5(3)$	$\Omega^* x^*$
$i(3,-)$	$Z_2 \times (S_4 \text{ wreath } Z_2)$	$\Omega^* x^*$
$i(1,-$	$Z_2 \times PO_5(3)$	$\Omega^* x^* p^*$
$i(3,+)$	$Z_2 \times (S_4 \text{ wreath } Z_2)$	$\Omega^* x^* p^*$
γ	$Z_2 \times \mathrm{Aut}(A_6)$	$\Omega^* \gamma$
γ^{x^*}	$Z_2 \times \mathrm{Aut}(A_6)$	$\Omega^* \gamma p^*$

TABLE 2.11

Involutions and Centralizers in $\mathrm{Aut}(\Omega_6^+(3))$

Involution	Centralizer	Coset
$i(4,+)$	$Z_2/Z_2 \times (E_4/SL_2(3)^* SL_2(3))$	Ω^*
$i(2,-)$	$Z_2/(S_6 \times D_8)$	Ω^*
$i(1,+)$	$Z_2 \times PO_5(3)$	$\Omega^* x^*$
$i(3,+)$	$Z_2 \times (S_4 \text{ wreath } Z_2)$	$\Omega^* x^*$
$i(1,-)$	$Z_2 \times PO_5(3)$	$\Omega^* x^*$
ρ^*	$Z_2 \times \mathrm{Aut}(L_3(3))$	$\Omega^* \rho^*$
ρ_0^*	$Z_2 \times \mathrm{Aut}(A_6)$	$\Omega^* \rho^* x^*$

of $O([V,z])$ and its action on $[V,z]$ are discussed in [3, pp. 400, 401]. In particular, there are $K_i \trianglelefteq\trianglelefteq C_G(z)$ with $z \in K_i \cong SL_2(3)$, $i=1,2$. Also

(2.12) *Let $x^* \in \mathscr{I}(\Gamma^*)$ act on $[V,z]$. Then*

(1) *x^* induces an outer automorphism on K_1^* and K_2^* precisely when x^* is of type $i(2,+)$ on $[V,z]$.*

(2) *x^* induces a nontrivial inner automorphism on K_1^* and K_2^* precisely when x^* is of type $i(2,-)$ on $[V,z]$.*

(3) *$K_1^* = K_2$ precisely when x^* is of type $i(1,\alpha)$ or $i(3,\alpha)$ on $[V,z]$.*

(4) *x^* induces an inner automorphism on K_i^* and a nontrivial outer automorphism on K_{3-i}^* precisely where $x^* \in \Gamma^* - G^*$.*

(2.13) *Let $n=8$, $\varepsilon=+$, s an involutory automorphism of Ω^* and either $X^* = E(C_{\Omega^*}(s))$ of s of type $i(3,\alpha)$ and $X^* = O_2(O^2(C_{\Omega^*}(s))$. Then*

(1) *If s is of type $i(6,\alpha)$ or $i(4,-)$ there is $u^* \in X^*$ of type $i(4,+)$ with $u^*s \in s^{\Omega^*}$.*

(2) *If s is of type $i(3,\alpha)$ or $i(1,\alpha)$ there is $u^* \in X^*$ of type $i(2,-)$ with $u^*s \in s^{\Omega^*}$.*

Proof. We will state a number of other results of this type in later lemmas; the proofs are all the same. We prove (1) for s of type $i(6,\alpha)$ and omit all the other proofs.

As $m([V,s])=6$ there is a subspace U of dimension 4 and sign $+$ of $[V,s]$. $O^2(O(U)) \leqslant X$, so we may take $z \in X$ with $[V,z]=U$. Then z is of type $i(4,+)$. $[V,sz]=C_{[V,s]}(z)$ is of sign $\mathrm{sgn}([V,z])\,\mathrm{sgn}(U)=\alpha$ and dimension $6-\dim U=2$, so sz is of type $i(2,\alpha)$. By 2.4, $i(2,\alpha)=i(6,\alpha)$, so $sz^* \in s^{\Omega^*}$.

Notice if s is of type $i(4,-)$ then $X^* = X_1^* \times X_2^*$, $X_1 = E(O([V,s])) \cong$

$L_2(9)$ and $X_2 = E(O(C_V(s))) \cong L_2(9)$. $\mathscr{I}(X_i) \subseteq i(2, -)$ and we choose u^* to be a diagonal involution in X_1X_2. Otherwise the proof is the same.

(2.14) *Let $n = 7$, s an involutory automorphism of Ω^*. Then either*

(1) *there is u^* of type $i(4, +)$ in $C_{\Omega^*}(s)$ with $u^*s \in s^{\Omega^*}$, or*

(2) *s is of type $i(6, \pm 1)$ and there is u^* of type $i(2, -)$ in $C_{\Omega^*}(s)$ with $u^*s \in s^{\Omega^*}$. There is an involution of type $i(4, +)$ in $u^*E(C_{\Omega^*}(s))$. $E(C_{\Omega^*}(\langle u^*, s\rangle)) \cong \Omega_5(3)$ and if v induces an involutory inner automorphism on Ω^* centralizing $s \in i(6, +)$ with $E(C_{\Omega^*}(\langle v, s\rangle)) \cong \Omega_5(3)$ then v is of type $i(2, -)$.*

(2.15) *Let $n = 6$, $\varepsilon = -$, and s an involution acting on Ω. Set $X = O^{2'}(O^2(C_\Omega(s)))$. Then*

(1) *If s induces an automorphisme of type p^*, γ, or $i(3, \alpha)$ on Ω^*, then there is u of type $i(4, +)$ in X with $us \in s^\Omega$.*

(2) *If s induces an automorphism of type $i(4, -)$ on Ω^* then $\mathscr{I}(X) \subseteq i(2, -)$ and if $u \in \mathscr{I}(X)$ then su induces an automorphism of type $i(4, -)$ on Ω^*.*

(3) *If s induces an automorphism in $\Gamma^* - \Omega^*$ then $\mathscr{I}(X) \subseteq i(4, +)$.*

(4) *If s is of type $i(3, \alpha)$ then $O_2(C_\Omega(s))$ is abelian.*

Proof. The proofs of (1) and (2) are analogous to those in 2.13. Notice that in (2), while su has the same type as s on Ω^*, since $Z(\Omega) = \langle \pi \rangle \neq 1$ it is not necessarily true that $su \in s^\Omega$. If s induces an automorphism in $\Gamma^* - \Omega^*$, then by 2.10, s is of type γ or γ^{x^*} and $\Phi(S) \leqslant X$, $S \varepsilon \mathrm{Syl}_2(C_\Omega(X))$. If $s \in C(z)$ then from a remark in the proof of 2.9, $z \in \Phi(R)$, $R \in \mathrm{Syl}_2(C_\Omega(\langle s, z\rangle))$, so (3) holds. If s is of type $i(3, \alpha)$ then from 2.6, $O_2(C_{\Omega^*}(s)) = X^*$ and $(X^*)^\# \subseteq i(4, +) \cup i(2, -)$. Thus all elements of $(X^*)^\#$ lift to involutions, so (4) holds.

(2.16) *Let $n = 6$, $\varepsilon = -$, $S^* \in \mathrm{Syl}_2(\Gamma^*)$. Then*

(1) *$S^* \cap \Omega^*$ has two E_{16}-subgroups B_i^*, $i = 1, 2$.*

(2) *B_1^* is conjugate to B_2^* in Γ^* but not in G^*.*

(3) *$\mathrm{Aut}_{\Omega^*}(B_i^*) \cong A_6$ and $\mathrm{Aut}_{\Gamma^*}(B_i^*) \cong S_6$.*

(4) *$C_{\Omega^*}(B_i^*) = B_i^*$ and $C_{G^*}(B_i^*) = A_i^* \cong E_{32}$. The elements of $A_i^* - B_i^*$ are of type $i(1, \alpha_i)$ and $i(3, -\alpha_i)$.*

(5) *A_i^* is the permutation module for $\mathrm{Aut}_{\Gamma^*}(B_i^*) \cong S_6$ modulo its one dimensional fixed space.*

(6) *If s^* is of type $i(3, \alpha)$ with $(K_1^*)^{s^*} = K_2^*$ then s^*z^* is of type $i(1, -\alpha)$.*

(7) *If* $s^* \in \mathscr{I}(\Gamma^*) - \mathscr{I}(\Omega^*)$ *there exists* $u^* \in \mathscr{I}(O^2(C_{\Omega^*}(s^*)))$ *with* $s^*u^* \in (s^*)^{\Omega^*}$.

(2.17) *Let* $n = 6$, $\varepsilon = +$, $s^* \in \mathscr{I}(\Gamma^*)$, *and* $X^* = O^{2'}(O^2(C_{\Omega^*}(s^*)))$. *Let* $S^* \in \mathrm{Syl}_2(\Gamma^*)$. *Then*

(1) *If* s^* *is not of type* $i(4, \pm 1)$ *or* $i(1, \pm 1)$ *then there is* u^* *of type* $i(4, +)$ *in* X^* *with* $s^*u^* \in (s^*)^{\Omega^*}$.

(2) *If* s^* *is of type* $i(4, -)$ *then* $\mathscr{I}(X^*) \subseteq i(2, -) = i(4, -)$.

(3) *If* $X^* \cong L_2(9)$ *and* $\mathscr{I}(X^*) \subseteq i(4, -)$ *then* s^* *is of type* $i(4, -)$.

(4) $S^* \cap \Omega^*$ *has two normal* E_{16}*-subgroups* B_i^*, $i = 1, 2$. B_1^* *is conjugate to* B_2^* *in* Γ^*, *but not in* G^*.

(5) $\mathrm{Aut}_{\Omega^*}(B_i^*) \cong S_5 \cong \mathrm{Aut}_{\Gamma^*}(B_i^*)$ *acts naturally on* B_i^*.

(6) $C_{\Omega^*}(B_i^*) = B_i^*$ *and* $C_{\Gamma^*}(B_i^*) = A_i^* \cong E_{32}$. $A_i^* = B_i^* \times \langle t_i^* \rangle$, t_i^* of type $i(1, \alpha_i)$, $t_i^* \in C_{G^*}(N_{\Omega^*}(B_i^*))$.

Proof. Straightforward.

(2.18) *Let* $n = 5$, *and* $S^* \in \mathrm{Syl}_2(G^*)$. *Then*

(1) $J(S^* \cap L^*) = B^* \cong E_{16}$.

(2) $\mathrm{Aut}_{\Omega^*}(B^*) \cong A_5$ *and* $\mathrm{Aut}_{G^*}(B^*) \cong S_5$ *act naturally on* B^*. $B^* = C_{G^*}(B^*)$.

(3) *If* $u^* \neq z^*$ *is of type* $i(4, +)$ *in* $C(z^*)$ *then* u^*z^* *is of type* $i(2, -)$.

(4) *If* u^* *is of type* $i(2, -)$ *in* $C(z^*)$ *then either* $(K_i^*)^{u^*} = K_i^*$ *and* $u^*z^* \in (u^*)^{K_i^*}$ *or* $(K_1^*)^{u^*} = K_2^*$ *and* u^*z^* *is type* $i(4, +)$.

(5) *If* u^* *is of type* $i(2, +)$ *and* $(K_1^*)^{u^*} = K_2^*$ *then* u^*z^* *is of type* $i(4, -)$.

(6) *If* s^* *is of type* $i(2, \alpha)$ *and* $u^* \in \mathscr{I}(G^*) - \{s^*\}$ *centralizes* $O^2(C(s^*))$ *then* u^* *is of type* $i(4, \pm 1)$.

(2.19) *Let* $H \cong \Omega_n^\varepsilon(3)$, $5 \leqslant n \leqslant 8$, z of type $i(4, +)$ in H, or $H \cong L_3^\varepsilon(3)$ *and* $z \in \mathscr{I}(H)$. *Let* $z \in K \cong SL_2(3) \trianglelefteq\trianglelefteq C_H(z)$, *and* t *an involutory automorphism of* H *with* $t \notin zC(H)$. *Then* $K \leqslant X \in \mathscr{C}(C_H(t))$, *and* $X \in \mathscr{F}^*$.

Proof. This follows by inspection of $\mathrm{Aut}(H)$.

(2.20) *Let* $H \cong \Omega_n^\varepsilon(3)$, $5 \leqslant n \leqslant 8$, $L_3^\varepsilon(3)$, *or* $L_2(3^m)$, $H \leqslant A \leqslant \mathrm{Aut}(H)$, *and* $a \in \mathscr{I}(A)$. *Then*

(1) $O_3(C_A(a)) = 1$.

(2) *If* $B \trianglelefteq\trianglelefteq C_A(a)$ *with* $B/O(B) \cong SL_2(3)$, *then* $a \in B$ *and either* a *is of type* $i(4, +)$ *in* $\cong \Omega_n^\varepsilon(3)$ *or* $a \in H \cong L_3^\varepsilon(3)$.

(3) *If* $X \in \mathscr{C}(C_H(a))$, $b \in \mathscr{I}(X)$, *and* $b \in B \trianglelefteq\trianglelefteq C_X(b)$ *with* $B/O(B) \cong SL_2(3)$, *then either* a *is of type* $i(4,+)$ *in* $H \cong \Omega_n^{\varepsilon}(3)$ *or* $a \in H \cong L_3^{\varepsilon}(3)$.

Proof. Again this is a consequence of the structure of $\mathrm{Aut}(H)$.

3

In this section and throughout the remainder of the paper, assume G is a group satisfying Hypothesis A with $F^*(G) \notin \mathrm{Chev}(3)$, $t \in \mathscr{I}(G)$, $L \in \mathscr{C}(C_G(t))$ with $L \in \mathscr{F}$, and $S \in \mathrm{Syl}_2(C_G(t))$. Then $Z(S) \cap L$ contains an involution z of type $i(4,+)$. This notation will be maintained throughout the remainder of the paper.

Following Walter in [15], define relations on $\mathscr{C} = \mathscr{C}(\mathscr{I}(G))$ as follows: Let $X, Y \in \mathscr{C}$ and write

$$X \to Y$$

respectively

$$X \to YY^x$$

if there exist $x, y \in \mathscr{I}(G)$ with $[x, y] = 1$, $X \in \mathscr{C}(C_G(x))$, $Y \in \mathscr{C}(C_G(y))$, and

$$X \leqslant Y = [Y, x]$$

respectively

$$X \leqslant YY^x \neq Y.$$

Further write

$$X \twoheadrightarrow Y$$

if there is a sequence

$$X = X_1, \dots, X_n = Y$$

such that $X_i \to X_{i+1}$ or $X_i \to X_{i+1}X_{i+1}^{x_i}$ for each $i = 1, \dots, n-1$. Let $\mathscr{C}^*$ consist of those $X \in \mathscr{C}$ such that $X \twoheadrightarrow Y$ implies $X/Z(X) \cong Y/Z(Y)$. In particular if $X \in \mathscr{C}^*$ and $Y \in \mathscr{C}$, then we cannot have $X \to Y$, while if $X \to YY^x$ then $Y \in \mathscr{C}^*$. Indeed

(3.1) *Let* $X \in \mathscr{C}$. *Then the following are equivalent*:

(1) X *is standard in* G.

(2) $X \in \mathscr{C}(C_G(x))$ *for each* $x \in \mathscr{I}(C_G(X))$.

(3) $X \in \mathscr{C}^*$.

Proof. See, for example, Proposition 1.2 in Walter [15].

(3.2) *Let $X, Y \in \mathscr{C}$ with $X \twoheadrightarrow Y$ and $X \in \mathscr{F}^*$. Then $Y \in \mathscr{F}^*$, and if $X \in \mathscr{F}$ then even $Y \in \mathscr{F}$.*

Proof. This is a consequence of Hypothesis A.

(3.3) *If $X \in \mathscr{C}^* \cap \mathscr{F}^*$ then $C_G(X)$ has cyclic Sylow 2-subgroups.*

Proof. By 3.1, X is standard in G. Then as $X \in \mathscr{F}^*$, if the lemma fails $F^*(G)$ is identified by Aschbacher–Seitz [4] or Corollary II in [3]. In either case as $L \in \mathscr{F} \cap \mathscr{C}$, we conclude $F^*(G) \in \mathrm{Chev}(3)$, contrary to the hypothesis of this section.

(3.4) *Let $x \in \mathscr{I}(G)$, $X \in \mathscr{C}(C_G(x)) \cap \mathscr{F}^*$, $y \in \mathscr{I}(X)$, $y \in K \trianglelefteq\trianglelefteq C_X(y)$ with $K/O(K) \cong SL_2(3)$, $Q = O_2(K)$, $u \in C(X\langle x\rangle)\, y$, and $I \in \mathscr{C}(C_G(u)) \cap \mathscr{F}^*$. Then*

(1) *either $[K, I] = 1$ or $Q \leqslant I$.*

(2) *If $Q \leqslant I$ and $[y, I] \neq 1$ then either $I \in \mathscr{F}$ and y is of type $i(4, +)$ in I or $I \cong L_n^\varepsilon(3)$.*

(3) *$\mathscr{C}(C_G(y)) \cap \mathscr{F}^*$ is empty.*

Proof. By Theorem 2 in [2], either $[Q, I] = 1$ or $Q \leqslant I$. Suppose $[Q, I] = 1 \neq [K, I]$ and set $I_0 = \langle I^{K\langle x\rangle}\rangle$ and $N(I_0)\,\alpha = N(I_0)/C(I_0)$. Then $1 \neq K\alpha \leqslant O_3(C(x\alpha))$, so a standard argument shows $K\langle x\rangle \leqslant N(I)$. But as $I \in \mathscr{F}^*$, $O_3(C(x\alpha)) = 1$ by 2.20.1, a contradiction. Therefore (1) holds.

Suppose $Q \leqslant I \not\leqslant C(y)$. $K\langle x\rangle \leqslant N(Q) \leqslant N(I)$. Then as $\mathrm{Aut}_K(I) \trianglelefteq\trianglelefteq C_{\mathrm{Aut}(I)}(x)$, (2) follows from 2.20.2.

Finally suppose $y = u$. If $Q \leqslant I$ then as $u \in Q$, $u \in Z(I)$, so as $I \in \mathscr{F}^*$, $I/O(I) \cong \Omega_6^-(3)/Z_2$. Now $A_4 \cong \mathrm{Aut}_K(I) \trianglelefteq\trianglelefteq C_{\mathrm{Aut}(I)}(x)$, so x is of type $i(3, \alpha)$ on I by 2.10. But then by 2.15.4, $O_2(C_I(x))$ is abelian, contradicting $Q \leqslant O_2(C_I(x))$.

So $[K, I] = 1$ by (1). Let $Y = II^x$ and $v \in \mathscr{I}(C_Y(x)\langle u\rangle)$ with $v \notin uC(X)$. Then $[K, v] = 1$ and $v \notin uC(X)$, so by 2.19, $K \leqslant J \in \mathscr{C}(C_X(v))$ with $J \in \mathscr{F}^*$. Let $J \leqslant DD^x$ with $D \in \mathscr{C}(C_G(v))$. Then $D \in \mathscr{F}^*$ by Hypothesis A.

Suppose $J \leqslant D$. Then by 2.20.3 we may replace x by v to assume $x \in Y\langle u\rangle$. Now as $[K, Y] = 1$, $K \trianglelefteq\trianglelefteq C_G(u)$, so by Corollary III in [3], $F^*(G) \in \mathrm{Chev}(3)$, contrary to the hypothesis of this section.

So $D \neq D^x$. Now there is $u_1 \in \mathscr{I}(D)$ with $u_1 \in K_1 \trianglelefteq\trianglelefteq C_D(u_1)$ and $K_1/O(K_1) \cong SL_2(3)$. Moreover $D^x \leqslant I_1 I_1^v$, $I_1 \in \mathscr{F}^* \cap \mathscr{C}(C_G(u_1))$. Replacing (x, X, u) by (v, D, u_1), we may assume $r \in \mathscr{I}(C(x) - N(X))$, $X^r \leqslant Y$, $v = u^r$, and $J = X$. Then $D = I^r$ and $[K^r, D] = 1$. Thus there is w of order 4 in $C_{K^r}(D^x\langle u_1\rangle)$ with $w^2 = v$. So as $D^x \leqslant I_1 I_1^v \cap C(w)$, w acts on $I_1 I_1^v$, and then

$v = w^2 \in N(I_1)$. Hence replacing (x, X, u, v) by (v, D^x, u_1^x, u_1), we have a contradiction as $v \in N(I_1)$.

This contradiction establishes (3) and completes the proof of the lemma.

(3.5) *Let $x \in \mathscr{I}(G)$, $X \in \mathscr{C}(C_G(x)) \cap \mathscr{F}^*$, $y \in \mathscr{I}(X)$, $y \in K \trianglelefteq\trianglelefteq C_X(y)$ with $K/O(K) \cong SL_2(3)$, and $u \in \mathscr{I}(C_G(y))$. Then $\mathscr{C}(C_G(\langle u, y\rangle)) \cap \mathscr{F}^*$ is empty.*

Proof. If not, by Hypothesis A there is $I \in \mathscr{C}(C_G(y)) \cap \mathscr{F}^*$, contrary to 3.4.3.

(3.6) *Let $x \in \mathscr{I}(G)$ and $X \in \mathscr{C}(C_G(x))$ with $X \in \mathscr{F}$* or $X \cong L_3^\varepsilon(3)$. *Then $\mathscr{C}(C_G(x)) \cap \mathscr{F}^* = \{X\}$.*

Proof. There is $y \in \mathscr{I}(X)$ with $y \in K \trianglelefteq\trianglelefteq C_X(y)$ and $K/O(K) \cong SL_2(3)$, so 3.5 completes the proof.

(3.7) *If $g \in G$ with $z^g \in C(t)$ then $E(C_L(z^g)) = 1$.*

Proof. This is a consequence of 3.5.

(3.8) *L is not isomorphic to $\Omega_8^-(3)$.*

Proof. If so $E(C_L(z)) \cong L_2(9)$, contrary to 3.7.

(3.9) *If $u \in \mathscr{I}(L)$ and $u \in K \in \mathscr{C}(C_L(u))$ with $K/O(K) \cong \Omega_6^-(3)/Z_2$, then $\{K\} = \mathscr{C}(C_G(u)) \cap \mathscr{F}^*$.*

Proof. $K \in \mathscr{C}(C_G(\langle u, t\rangle))$, so by Hypothesis A and 3.6, $K \leqslant I \in \mathscr{C}(C_G(u))$. As $u \in K$, $O_2(I) \neq 1$, so $I/O(I) \cong \Omega_6^-(3)/Z_2 \cong K/O(K)$. Hence $K = I$, and 3.6 completes the proof.

(3.10) *Assume $L \cong \Omega_8^+(3)$. Then $tz \in t^G$.*

Proof. Assume not. By 3.2, $L \in \mathscr{C}^*$. By the Z^*-theorem there is $t \neq s = t^g \in C(t)$. As $L \in \mathscr{C}^*$, $L = [L, s]$ by 3.3. It suffices to show there is $x \in (z^g)^{L^g}$ with $sx \in s^L$, so assume not. By 3.4.3, $s \notin z^L$ and we may assume $s \notin (tz)^L$, so by 2.8 we may assume s induces an automorphism of type $i(6, \alpha)$, $i(4, -)$, $i(1, \alpha)$, or $i(3, \alpha)$ on L.

Set $X = E(C_L(s))$. If s is not of type $i(4, -)$ we may assume $z \in K \trianglelefteq\trianglelefteq C_X(z)$ with $K/O(K) \cong SL_2(3)$, so by 2.20.3, z is of type $i(4, +)$ in L; that is $z \in (z^g)^{L^g}$. If s is of type $i(4, -)$ then $X = X_1 \times X_2$, $X_i \cong L_2(9)$, and we may take z to be a diagonal involution of X. $X_i \in \mathscr{C}(C_{L^g}(t))$, so t is of type $i(4, -)$ on L and again we conclude $z \in (z^g)^{L^g}$.

Next by 2.13.1 if s of type $i(6, \alpha)$ or $i(4, -)$ then $sz \in s^L$, as desired. Hence no member of $t^G \cap C(t)$ induces such an automorphism. If s is of type $i(1, \alpha)$

or $i(3, \alpha)$ then by 2.13.2 there is u of type $i(2, -) = i(6, -)$ in X with $us \in s^L$. As $X = E(C_{L^g}(t))$, t has the same type on L^g as s does on L (or a conjugate of this type under $\mathrm{Aut}(L^g)$). Hence u is of type $i(6, -)$ in L^g, so us has this type on L^g. As $us \in s^L$, this contradicts an earlier remark.

We can now show L is not isomorphic to $\Omega_8^+(3)$ using an argument of John Walter in [15]:

(3.11) *L is not isomorphic to $\Omega_8^+(3)$.*

Proof. Assume otherwise. By 3.10, $tz = z^g$ for some $g \in G$. There are $K_i \trianglelefteq\trianglelefteq C_L(z)$, $z \in K_i \cong SL_2(3)$, $1 \leqslant i \leqslant 4$. Set $Q_i = O_2(K_i)$ and $Q = \prod_i Q_i$. By 3.4.1 and 3.3, $Q \leqslant L^g$ and by 3.4.2, z is of type $i(4, +)$ in L^g. This forces $Q = O_2(C_{L^g}(z))$ and as $KO(C(t)) = O^2(C_G(\langle t, z\rangle)) = O^2(C_{L^g}(z))\, O(C(tz))$, $C_{L^g}(z) = X$ permutes the groups Q_i. In particular X acts on the set Δ of involutions in Q centralizing exactly two Q_i. The elements of Δ are of type $i(6, -)$ in L, conjugates of such involutions under $\mathrm{Aut}(L)$, and the analogous statement is also true for L^g. Let $d \in \Delta$. By 3.9,

$$L_d = E(C_L(d)) = E(C_{L^g}(d)).$$

But now, $L = \langle L_d : d \in \Delta\rangle = L^g$, contradicting $t^g = tz \notin C(L)$.

4

In this section we assume $L/O(L) \cong \Omega_7(3)$. By 3.2, 3.8, and 3.11,

(4.1) $L \in \mathscr{C}^*$.

So by 3.3,

(4.2) *$C(L)$ has cyclic Sylow 2-groups.*

(4.3) $z^G \cap C(t) \subseteq z^L \cup (tz)^L$.

Proof. From Section 2, most particularly 2.2, 2.6, and 2.7, $E(C_L(s)) \neq 1$ for each $s \in \mathscr{I}(C(t)) - (z^L \cup (tz)^L)$. Hence 3.7 completes the proof.

(4.4) (1) $\mathscr{A}(S \cap LC(L)) = \{A\}$.

(2) $m(A) = 7$.

(3) *$A \cap L$ is the natural module for $N_L(A)/A \cong A_7$.*

(4) $A \cap L = \langle^{N_L(A)}\rangle$.

Proof. From the structure of $\mathrm{Aut}(L)$ there is $E_{64} \cong B \leqslant L$ with

$B = C_L(B)$ and $\mathrm{Aut}_L(B) \cong A_7$ acting naturally on B. The Lemma can be easily derived from these observations and 4.2.

(4.5) *Either*

(1) $tz \in t^G$, *or*

(2) $t^G \cap C(t) = \{t\} \cup (tu)^L$, *where* $u \in \mathscr{I}(L)$ *is of type* $i(6, -)$.

Proof. Assume otherwise. By the Z^*-theorem there is $s = t^g \in C(t) - \{t\}$. By 2.14 either there is $v \in z^L$ with $sv \in s^L$, or s is of type $i(6, \alpha)$ on L. Assume the former. By 4.3 there is $w \in (z^g)^{L^g}$ with $v = w$ or sw. Hence either $sw \in s^L$ or $w \in s^L$. In the first case (1) holds; the second is impossible by 4.3.

So s is of type $i(6, \alpha)$. We may assume (2) fails and hence by 3.9 take s of type $i(6, +)$ on L. By 2.14 there is v of type $i(2, -)$ in $C_L(s)$ with $sv \in s^L$ and w of type $i(4, +)$ in $vE(C_L(s))$. $E(C_L(s)) \leqslant L^g$, so if $v \notin L^g$ then $w \notin L^g$. The latter is impossible by 4.3, so $v \in L^g$. Now $sv \in s^L$ induces an inner automorphism on L^g with

$$E(C_{L^g}(\langle sv, t\rangle)) = E(C_L(\langle s, v\rangle)) \cong \Omega_5(3),$$

and $E(C_{L^g}(t)) \cong E(C_L(s)) \cong \Omega_6^+(3)$, so t is of type $i(6, +$ on L^g. Hence by 2.14, sv is of type $i(2, -)$, so (1) holds by the first paragraph.

(4.6) (1) Either $tz \in z^G$ *and* $A = \langle z^G \cap A\rangle$ *or* $z^G \cap C(t) = z^L$ *and* $A \cap L = \langle z^G \cap A\rangle$.

(2) *A is weakly closed in S with respect to G, so* $t^G \cap A = t^{N(A)}$.

Proof. Part (1) is a consequence of 4.3 and 4.4.4. If $A^g \leqslant S$ then $B = \langle A^g \cap z^G\rangle \leqslant LC(L) \cap S$ by 4.3, so by (1) and 4.4.1, either $A^g = B = A$ or $B = (A \cap L)^g = A \cap L$. As $\langle t\rangle = \Omega_1(C_S(L))$, in either case $A = A^g$.

Set $M = N_G(A)$ and $\Delta = t^M$.

(4.7) $tz \in t^G$.

Proof. Assume not. By 4.5, $t^G \cap C(t) = \{t\} \cup (tu)^L$, where $u \in \mathscr{I}(L)$ is of type $i(6, -)$. Choose $u \in Z(S)$. Then by 4.6

$$t^G \cap A = \Delta = \{t\} \cup (tu)^{(M \cap L)}.$$

By 4.4, $(tu)^{M \cap L}$ is or order 7, so Δ is or order 8. Also $C_M(t)^\Delta = S_7$ or A_7, so $M^\Delta = S_8$ or A_8. $M_\Delta = C_M(t)_\Delta = C_G(A)$. Further $C_S(A) = C_S(L)A$, so $\Omega_1(\Phi(C_S(A))) \leqslant \langle t\rangle$, and then as $M \nleqslant C(t)$, $C_S(L) = \langle t\rangle$ and $A = C_S(A)$. Thus $C_G(A) = O(C(t))A = O(M)A$ and $M^\Delta \cong M/O(M)A \cong S_8$ or A_8.

Moreover A is the permutation module of degree 8 for M^{Δ}, modulo its 1-dimensional fixed space. In particular $B = A \cap L = [A, M \cap L] = [A, M] \trianglelefteq M$. $M \cap L$ splits over B, so $M/BO(M)$ is not the covering group of A_8 or S_8. Thus M splits over $BO(M)$ if $C_M(t)$ splits over $BO(M)$. If the latter fails there is $x \in S$ with $x^2 = t$ inducing an automorphism of type $i(6, +)$ on L. We may choose $E(C_L(\langle x, u \rangle)) \cong \Omega_5(3)$. $C_L(u) = E(C_L(u))\langle v \rangle$, v of type $i(4, +)$ in L, and by 4.3, v induces an inner automorphism on L^g. Thus $\langle t, C_L(u) \rangle = \langle tu, C_{L^g}(u) \rangle$, where $t^g = tu$, so x induces an outer automorphism on L^g. Now $\langle x, t, u \rangle$ is determined up to conjugacy under $E(C_L(u)) = E(C_{L^g}(u))$, so $\langle x, t, u \rangle \in (\langle x, t, u \rangle^g)^{L^g}$. But then $\langle tu \rangle = \Phi(\langle x, t, u \rangle^g) = \Phi(\langle x, t, u \rangle) = \langle t \rangle$, a contradiction.

So M splits over $BO(M)$ and hence there is a subgroup M_0 of M of index 2 which does not contain t. Let $s = t^h \in M$. Claim $s \in A$. For it not, from the action of M^{Δ} on A there is $a \in A$ such that $[s, a] = v$ has support of order 4 in the basis of the permutation module; that is to say $v \in z^M$. Then $sv \in s^G$, which is impossible by 4.3 since neither tz nor z is in t^G.

We have shown $t^G \cap M = \Delta$. Thus M contains a Sylow 2-group of G and as $\Delta \cap M_0$ is empty, $t \notin O^2(G)$ by Thompson transfer. But $u \in O^2(G)$ and by 3.9, $\mathscr{C}(C_G(u))$ has a member in $\mathscr{F}$. Hence by induction on the order of G, $F^*(G) \in \mathrm{Chev}(3)$, contrary to the hypothesis of this section.

(4.8) (1) $z^G \cap C(t) = z^L$.

(2) $B = A \cap L = \langle z^G \cap A \rangle \trianglelefteq M$.

Proof. Part (1) is a consequence of 4.3 and 4.7, while (1) and 4.6 imply (2).

(4.9) *Let* $u \in \mathscr{I}(B)$ *be of type* $i(6, -)$. *Then either*

(1) $C_M(u) \leqslant C_M(t)$, *or*

(2) $tu \in t^G$ *and* $C_M(u) \leqslant N_M(\langle u, t \rangle)$.

Proof. Let $K = E(C_L(u))$. By 3.9, $K \trianglelefteq C_G(u)$. $\langle u, t \rangle = D = C_A(K)$, so $C_M(u) \leqslant N(D)$ and the lemma holds.

By 4.4, $C_M(t)$ has three orbits Γ_i, $1 \leqslant i \leqslant 3$, on $B^{\#}$ of length 7, 21, and 35, with representative u_i of type $i(6, -)$, $i(2, -)$, and $i(4, +)$, respectively. By 4.8.2, $\Delta \subseteq tB$, so

$$\Delta - \{t\} = \bigcup_{i \in I} t\Gamma_i$$

for some $I \subseteq \{1, 2, 3\}$. By 4.7, $3 \in I$. Thus Δ is of order 36, 43, 57, or 64, for I equal to $\{3\}$, $\{1, 3\}$, $\{2, 3\}$, or $\{1, 2, 3\}$, respectively.

Next M^{Δ} is a section of $GL(A) \cong L_7(2)$. In particular its order is not divisible by 43 or 57, so Δ is or order 36 or 64. Let $u = u_2$ and $\Gamma = u^M$. By

4.8, $z \notin \Gamma$, so $\Gamma = \Gamma_2$ or $\Gamma_1 \cup \Gamma_2$ is of order 7 or 28. On the other hand, by 4.9, $|C_M(u): C_M(\langle u, t \rangle)| \leqslant 2$, so $|C_M(u)|_3 \leqslant |C_M(t)|_3$ and $|C_M(u)|_2 \leqslant 2\,|C_M(t)|_2$. But if Δ is of order 36 then as $|\Gamma|_3 = 1$, $|M|_3 = |C_M(u)|_3 \leqslant |C_M(t)|_3$, a contradiction. Similarly $|M|_2 \leqslant 4\,|C_M(u)|_2 \leqslant 8\,|C_M(t)|_2$, so Δ is not of order 64.

We have derived a contradiction from the hypothesis that $L/O(L) \cong \Omega_7(3)$. Thus we have established

THEOREM 4.10. *If G satisfies Hypothesis A, $F^*(G) \notin \mathrm{Chev}(3)$, and $L \in \mathscr{C}(\mathscr{T}(G))$, then $L/O(L)$ is not isomorphic to $\Omega_7(3)$.*

5

In this section we assume $L/O(L) \cong \Omega_6^-(3)/Z_2$. From results in previous sections we conclude

(5.1) $L \in \mathscr{C}^*$.

(5.2) *$C(L)$ has cyclic Sylow 2-subgroups, so $\langle t \rangle = O_2(L)$.*

(5.3) $\langle t, z \rangle = \Omega_1(Z(S))$.

(5.4) *If $z^g \in C(t)$ then either $z^g \in z^L$ or z^g induces an automorphism of type $i(3, \alpha)$ on L.*

Proof. By 2.10 and 3.7, $z^g \in z^L$ or $(zt)^L$, or z^g induces an automorphism of type $i(3, \alpha)$ on L. If $zt \in z^G$ then by 5.3, $S \in \mathrm{Syl}_2(G)$ and $zt \in z^{(N(S) \cap C(t))}$. This is impossible as $zt \notin z^{C(t)}$.

(5.5) *Either*

(1) *$tz \in t^G$, or*

(2) *each member of $t^G \cap C(t) - \{t\}$ induces an automorphism of type $i(1, \pm 1)$ on L.*

Proof. Assume otherwise. By the Z^*-theorem there is $s = t^g \in C(t) - \{t\}$. Set $X = E(C_L(s))$ unless s is of type $i(3, \alpha)$, where $X = O_2(O^2(C_L(s)))$. By 5.4, $s \notin z^L$ and by hypothesis $s \notin (tz)^L$. By Theorem 2 in [2] and 5.2, $X \leqslant L^g$. Suppose s is of type p^*, γ, or $i(3, \alpha)$, in the notation of 2.10. Then by 2.15.1 we may take $z \in X$ and $sz \in s^L$. But $z \in X \cap z^G \subseteq L^g \cap z^G = (z^g)^{L^g}$, contradicting $tz \notin z^L$. Hence by 2.10, s is of type $i(4, -)$. Then by 2.15.2, we may take $tz \in X$, so that $tz \in (tz)^G \cap L^g = ((tz)^g)^{L^g}$; that is, tz is of type $i(2, -)$ in L^g. $L_2(9) \cong X = E(C_{L^g}(t))$ and the involutions in X are of type

$i(2,-)$ in L^g, so by 2.10 and 2.15.3, t is of type $i(4,-)$ on L^g. Then by 2.15.2, $z = t \cdot tz$ is also of type $i(4,-)$ on L, contradicting 5.4.

(5.6) *Some member of t^G induces an automorphism of type $i(1,\alpha)$ on L.*

Proof. If not by 5.5, $tz = t^g$ for some $g \in G$. Let $E_{32} \cong B \leqslant S \cap L$. By 2.16, $B = \langle z^G \cap B \rangle$, so if $B \nleqslant L^g$ there is $b \in z^G \cap B - L^g$. By 5.4, b is of type $i(3,\alpha)$ on L^g. Also $tb \in t^G$ and $z \in O^2(C_L(z)) \leqslant L^g$, so $tb = (tz)(zb) \in bL^g$. Hence as L^g is transitive on the involutions of type $i(3,\alpha)$ and all other involutions in bL^g induce automorphisms on L of type $i(1,-\alpha)$, some member of t^G is of type $i(1,-\alpha)$ on L^g, contrary to assumption.

By 5.6 there is $r = t^g$ inducing an automorphism of type $i(1,\alpha)$ on L. By 2.16 there is $E_{32} \cong B \leqslant C_L(r)$ and $B = \langle t \rangle \times (E(C_L(r)) \cap B)$ with $E(C_L(r)) \cong \Omega_5(3)$. Set $A = \langle r, B \rangle$. $E(C_L(r)) \leqslant L^g$, so

(5.7) $A = (A \cap L)\langle r \rangle = (A \cap L^g)\langle t \rangle$.

Set $M = N_G(A)$ and $M^* = M/C_M(A)$.

(5.8) (1) $C_M(A) = AO(C(t)) = AO(M)$.

(2) *M is irreducible on A.*

(3) *$C_M(t)^*/C_M(L)^* \cong A_6$ or S_6 and $(L \cap M)^* \cong A_6$.*

(4) *A is the permutation module for $(L \cap M)^*$.*

(5) *The orbits of $L \cap M$ on $A^\#$ are described in Table* 5.8.

(6) *$z^M = z^{(M \cap L)} \cup (z')^{(M \cap L)}$ is of length* 35.

Proof. Parts (3) and (4) follow from 2.16. Part (4) and an easy calculation establish (5). $\langle t \rangle$, B, and A are the only nontrivial $(M \cap L)$-invariant subspaces of A, so by 5.7 either (2) holds or $B = A \cap L^g$. But then $r \in B \leqslant L$, a contradiction. $C_S(A) = C_S(L)A$ with $\Omega_1(\Phi(C_S(A))) \leqslant \langle t \rangle$, so (2) implies (1). Finally by (2), $B \neq A \cap L^g$, so there is $a \in z^G \cap A \cap L^g - B$. By 5.4, $z^G \cap A \subseteq z^{(M \cap L)} \cup (z')^{(M \cap L)}$, so (6) holds.

TABLE 5.8

Support	1	2	3	4	5	6
Length	6	15	20	15	6	1
Representative	r	tz	z'	z	rt	t
In L?	No	Yes	No	Yes	No	Yes

(5.9) *Either*

(1) $t^G \cap A = \{t\} \cup r^{(M \cap L)} = t^M$ *is or order* 7 *and* $M^* \cong A_7$ *or* S_7 *acts naturally on* A, *or*

(2) $t^G \cap A = A^{\#} - z^M$ *is or order* 28 *and* $M^* \cong A_8$ *or* S_8 *acts naturally on* A.

Proof. M acts on $\Gamma = A^{\#} - z^M$ and Γ is the union of four $L \cap M$ orbits Γ_i, $i = 1, 2, 5, 6$, where Γ_i is described in column i of Table 5.8. Let $\Delta = t^M$. Then Γ_1 and Γ_6 are contained in Δ, so Δ is of order 7, 22, 13, or 28. The second and third cases are impossible as M^* is a subgroup of $GL_6(2)$, whose order is not divisible by 11 or 13.

Suppose Δ is or order 7. Then $\Delta = \{t\} \cup r^{(M \cap L)}$ and as $A_6 \cong (M \cap L)^*$ is maximal in A_7, $M^* \cong A_7$ or S_7 acts naturally on A. If $t^G \cap A \neq \Delta$ then $rt = t^h$ for some $h \in G$. But then we have symmetry between r and rt, so $(M \cap L^h)^* \cong A_6$, impossible as $E(C_M(rt)^*) \cong A_5$.

So assume Δ is of order 28. $(M \cap L)^* \leqslant X \leqslant GL(A)$ with $X \cong S_8 \cong O_6^+(2)$ acting naturally on A with orbits z^M and Δ on $A^{\#}$. This action induces a quadratic form f and a bilinear form (,) on A defined by $f(a) = 0$ if $a \in z^M, f(a) = 1$ if $a \in \Delta$, $(a, b) = 0$ if $a \in z^M$ and $ab \in b^M$ or if $a, b \in \Delta$ and $a, b \in z^M$. As this structure depends only on the partition $A^{\#} = \Delta + z^M$, it is preserved by M^*, and hence $M^* \leqslant X$. Then as $|M^*| \geqslant |X|/4$, $M^* \cong A_8$ or S_8 acts naturally on A.

(5.10) $M^* \cong A_8$ *or* S_8.

Proof. If not, by 5.9, $M^* \cong A_7$ or S_7. By 5.9 and 5.3, t is weakly closed in $Z(S)$, so $S \in \mathrm{Syl}_2(G)$. Let $K_i \trianglelefteq\trianglelefteq C_L(z)$, $i = 1, 2$, with $K_i/O(K_i) \cong SL_2(3)$, $Q_i = S \cap K_i$, $Q = Q_1 Q_2$, $R = C_S(Q)$, $H = C_G(z)$, $H\alpha = H/\langle z \rangle$, and X and T the strong closure of t in $C(Q)$ and R with respect to H, respectively. As $O_2(C_M(t)^*) = 1$, $\langle t \rangle = C_S(L)$ by 5.8.

First $R\alpha \cong D_8$ or E_4 and either $T = R$ or $R\alpha \cong D_8$ and $T\alpha \cong E_4$. Thus $\mathcal{J}(T\alpha) \subseteq (t\alpha)^H$ and $T\alpha \in \mathrm{Syl}_2(X\alpha)$ by Thompson transfer. Also if $h \in H$ with $(X\alpha) \cap (X\alpha)^h$ of even order then we may take $t^h \in T$. Hence as $[R, K_i] = 1$, $Q^h = O_2(O^2(C_H(t^h))) = Q$, so $h \in N(X)$. Therefore $X\alpha$ is tightly embedded in $H\alpha$.

Next if $t^h \in S - R$ then by 5.5 and 5.9, $t^h \in C(R) - N(K_1)$. As $[t^h, R] = 1$ we may take $T^h \leqslant C_S(R)$. This is impossible as $(T\alpha)^h$ contains a 4-group while each involution in $(T\alpha)^h$ moves $(K_1)\,\alpha$. Therefore $T\alpha$ is strongly closed in $S\alpha$ with respect to H, so as $X\alpha$ is tightly embedded in $H\alpha$, $T\alpha$ is Sylow in $Y\alpha = \langle (X\alpha)^H \rangle$. As $T\alpha \cong D_8$ or E_4 is Sylow in $Y\alpha$ and $[K_i, T\alpha] = 1$, $[K_i, Y] \leqslant O(Y)$, and $K_i O(H) \trianglelefteq\trianglelefteq H$. Hence $F^*(G) \in \mathrm{Chev}(3)$ by Corollary III in [3], contrary to the hypothesis of this section.

(5.11) *Either*

(1) $C_S(L)=\langle t\rangle$, $M^*\cong A_8$, *and* $C_M(t)^*\cong S_6$, *or*

(2) $C_S(L)\cong Z_4$, $[C_S(L),r]=\langle t\rangle$, $M^*\cong S_8$, *and* $C_M(t)^*\cong Z_2\times S_6$.

Proof. $[C_S(L),B]=1$, so $[C_S(L),A]=[C_S(L),r]\leqslant C_A(L)=\langle t\rangle$. By 5.8, $A=C_S(A)$, so $|C_S(L)|\leqslant 4$ with $[C_S(L),r]=\langle t\rangle$ in case of equality. Finally $M^*\cong A_8$ or S_8 with $C_M(t)^*\cong S_6$ or $Z_2\times S_6$, respectively, so 5.8.3 completes the proof.

(5.12) *Let* $U=O(C_G(t))$. *Then*

(1) *if* $U\neq 1$, *we have* $\langle L^{C(U)}\rangle\cong\Omega_8^+(3)$.

(2) $\langle t\rangle=Z(L)$.

Proof. By 5.9, $tz=t^h$ for some $h\in M$. Then

$$U\leqslant C(O_2(O^2(C_G(\langle t,z\rangle))))\leqslant C(L^h)$$

so $U=U^h$. Thus if $U\neq 1$ then by induction on the order of G, $\langle L,L^h\rangle=\langle L^{C(U)}\rangle\cong\Omega_8^+(3)$. In particular $U\cap L\neq 1$, so (2) holds.

By 5.12.2 there is $K_i\trianglelefteq\trianglelefteq C_L(z)$, $i=1,2$, with $K_i\cong SL_2(3)$. There is $h\in M$ with $t^h\in K_1K_2$ and $h^2\in C(t)$. Set $K_3=K_1^h$ and $K_4=K_2^h$. Set $\Delta(t)=\mathscr{I}(K_3K_4)-\{z\}$. Then K_3K_4 is transitive on $\Delta(t)$ and $t\in\Delta(t)$. Set $K(t)=K_1K_2=O^2(C_L(z))$. If $x\in C(z)$ with $t^x\in C_{\Delta(t)}(t)$ then by Theorem 2 in [2] and 5.12, $K(t)=O^2(C_{L^x}(z))=K(t^x)=K(t)^x$. Hence as the involution graph on $\Delta(t)$ is connected we conclude

(5.13) $K(t)=K(t^x)=K(t)^x$ *for each* $x\in C(z)$ *with* $t^x\in\Delta(t)$.

Now $K(t^h)=K_3K_4\leqslant N(K(t))$ so $[K(t),K(t^h)]\leqslant K(t)\cap K(t^h)=\langle z\rangle$, and hence as $K(t)=O^2(K(t))$, even

(5.14) $[K(t),K(t^h)]=1$, *so* $K=K(t)\,K(t^h)$ *is the central product of four copies of* $SL_2(3)$.

Set $Q_i=O_2(K_i)$ and $Q=O_2(K)$. Let $\Gamma=\{K_i\colon 1\leqslant i\leqslant 4\}$ and Δ the set of involutions contained in the product of two members of Γ. For $d\in\Delta$ let $K(d)$ be the product of two members of Γ centralizing d.

(5.15) *Let* $d\in\Delta$. *Then*

(1) $E(C_G(d))\cong\Omega_6^-(3)/Z_2$.

(2) $K(d)=O^2(C(z)\cap E(C_G(d)))$.

(3) *We have symmetry between* t *and* d.

Proof. If $d \in t^{C(z)}$ this is clear. Thus we may take $d \in C(t)$ and $K(d) = K_1 K_3$. Thus by 2.10, $E(C_L(d)) \cong E(C_{L^h}(d)) \cong U_3(3)$. By Hypothesis A and 3.6, $\langle E(C_L(d)), E(C_{L^h}(d))\rangle \leqslant I \in \mathscr{C}(C_G(d)) \cap \mathscr{F}$. As $E(C_I(t) \cong U_3(3)$, $I/Z(I) \cong \Omega_6^-(3)$. If $d \notin Z(I)$ then z is fused in I to a member of Δ, contradicting 3.4.3. Thus the lemma holds.

Let $H = C_G(z)$ and $H\alpha = H/\langle z\rangle$. Set $N = N_G(Q)$ and $N\beta = N/C_N(Q\alpha)$.

(5.16) (1) $QO(C(t)) = C_N(Q\alpha)$.

(2) *If* $x \in C_S(L) - \langle t\rangle$ *then* $K_3^x = K_4$.

(3) $C_N(Q_1 Q_2 \alpha) = QK(t^h)\, C_G(L)$.

Proof. If $x \in C_S(L) - \langle t\rangle$ then $\langle x^*\rangle = O_2(C_M(t)^*)$ and $[x, C_M(t)] \leqslant \langle t\rangle$, which allows us to calculate in M to determine (2). By 5.11 and 2.6, $|C(t) \cap C(Q_1 Q_2)|_2 = 2^{4+\varepsilon}$, where $2^{1+\varepsilon} = |C_S(L)|$. But $|Q_3 Q_4 C_S(L) \cap C(t)| = 2^{4+\varepsilon}$, so $C_H(Q) = \langle z\rangle\, O(C(t))$. As $C_H(Q\alpha) = QC_H(Q)$, (1) holds. Similarly (3) holds.

(5.17) *$N_G(Q)$ permutes Γ transitively.*

Proof. $\mathscr{I}(Q) = \Delta + \Delta'$, where Δ' consists of z together with the involutions centralizing no member of Γ. z is fused into $\Delta' - \{z\}$ under M and K is transitive on $\Delta' - \{z\}$, so $\Delta' = z^G \cap Q$. Hence $N_G(Q)$ acts on Δ. Then by 5.15, $N_G(Q)$ acts on $\{K(d): d \in \Delta\}$ and hence on Γ. $N_M(Q)$ is transitive on Γ.

(5.18) *Let* $1 \neq E_{2^n} \cong U\beta \leqslant N\beta$. *Then*

(1) $m([Q\alpha, u\beta]) > n$ *for some* $u\beta \in (U\beta)^{\#}$.

(2) $m(N\beta) \leqslant 3$.

Proof. Suppose $m([Q\alpha, u\beta]) < 4$ for each $u\beta \in (U\beta)^{\#}$. Then u has at most one nontrivial orbit on Γ, so because of 5.15.3 we may assume U fixes K_1 and K_2. Then by 5.16.2, U_β is faithful on $(Q_3 Q_4)\,\alpha$, so $n \leqslant m(\mathrm{Out}(Q_3 Q_4)) = 2$. Now if $m([Q\alpha, u\beta]) \leqslant n$ then for each $u\beta \in (U\beta)^{\#}$ then we may choose u with $[(Q_1 Q_2)\,\alpha, u\beta] = 1$. Then by 5.16, $K_3^u = K_4$, so $n = 2$ and $m([(Q_3 Q_4)\,\alpha, v\beta]) = 2$ for each $v\beta \in U\beta - \langle u\beta\rangle$. But now $m([Q\alpha, v\beta]) = 3$.

Thus (2) implies (1), so we may assume $n = 4$. Let V be the kernel of the action of U on Γ. $m(N^\Gamma) \leqslant 2$, so $m(V\beta) \geqslant 2$. By 5.16, an element of $(V\beta)^{\#}$ centralizes at most one $(Q_i)\,\alpha$, so $m(V\beta) = 2$ and U fixes each $(Q_i)\,\alpha$ centralized by some element of $(V\beta)^{\#}$. But as $m(V\beta) = 2$, $m(U^\Gamma) = 2$, so U fixes no member of Γ, a contradiction.

(5.19) *$Q\alpha$ is strongly closed in $N\alpha$ with respect to H.*

Proof. By 5.18.1, $Q\alpha = J(N\alpha)$, so $Q\alpha$ is weakly closed in $N\alpha$. Then 5.18.1 and Corollary 4 in [8] complete the proof.

(5.20) $F^*(G) \in \mathrm{Chev}(3)$.

Proof. By 5.19, the structure of $N\beta$, and the main theorem of [8], $(K_i)\alpha \leqslant (L_i)\alpha \trianglelefteq\trianglelefteq H\alpha$ with $(L_i)\alpha \cong L_2(q)$, $q \equiv \pm 3 \bmod 8$. Then $L_i \cong SL_2(q)$, so Corollary III in [3] completes the proof.

6

In this section we continue the hypothesis and notation of Section 3.

(6.1) *If* $L \cong \Omega_6^+(3)$ *and* $x \in \mathscr{I}(C(t))$ *with* $E(C_L(x)) \cong \Omega_5(3)$, *then* $E(C_L(x)) \leqslant I \in \mathscr{C}(C_G(x))$ *with* $I \cong \Omega_5(3)$ *or* $\Omega_6^+(3)$.

Proof. $X = E(C_L(x)) \in \mathscr{C}(C_G(\langle x, t\rangle))$, so by Hypothesis A and 3.6, $X \leqslant I \in \mathscr{C}(C_G(x)) \cap \mathscr{F}^*$. Then by results in previous sections, either the lemma holds or $I/O(I) \cong \Omega_6^-(3)$, and we may assume the latter. Let z be of type $i(4, +)$ in X. Then as $I/O(I) \cong \Omega_6^-(3)$, z is fused under I to u of type $i(2, -)$ in X. But as $L \cong \Omega_6^+(3)$, $E(C_L(u)) \cong L_2(9)$, contradicting 3.7.

HYPOTHESIS 6.2. *If* $X \in \mathscr{C}(\mathscr{I}(G))$ *with* $X \cong \Omega_6^+(3)$ *then* $\mathrm{Aut}_G(X) \cap PO_6^+(3) = \mathrm{Inn}(X)$.

(6.3) *Assume Hypothesis* 6.2 *and let* $x \in \mathscr{I}(G)$. *Then* $\mathscr{C}(C_G(x)) \cap \mathscr{F}^*$ *contains at most one member.*

Proof. Assume X_1 and X_2 are distinct members of $\mathscr{C}(C_G(x)) \cap \mathscr{F}^*$. By 3.6, $X_i \cong L_2(3^{n_i})$, $i = 2$ or 4. By 3.3, $X_i \notin \mathscr{C}^*$, so $n_i = 2$. Also there exists a series

$$X_1 = X = Y_1, \ldots, Y_n = Y$$

such that $Y \in \mathscr{C}^*$ and $Y_i \to Y_{i+1}$ or $Y_i \to Y_{i+1} Y_{i+1}^{x_i}$, for $i = 1, \ldots, n-1$. Choose j maximal such that $\mathscr{C}(C_G(y)) \cap \mathscr{F}^*$ has two or more members for some $y \in \mathscr{I}(C(X_j))$. Without loss $j = 1$. Hence $X_i \to X_{i+1}$ for $1 \leqslant i \leqslant n$ by maximality of j. In particular, $X \leqslant Y$. As $Y \in \mathscr{C}^*$ and $X \twoheadrightarrow Y$, $Y/O(Y) \cong \Omega_6^\varepsilon(3)$, $\Omega_5(3)$, or $L_2(81)$. Suppose $Y/O(Y) \cong \Omega_6^-(3)$. Then each $z \in \mathscr{I}(Y)$ is of type $i(4, +)$. In particular we may take $z \in X$. Now 3.5 supplies a contradiction.

So $Y \cong \Omega_6^+(3)$, $\Omega_5(3)$, or $L_2(81)$. In particular with Section 2 and Hypothesis 6.2, $X = E(C_Y(x))$ and taking $Y \in \mathscr{C}(C_G(y))$, $y \in \mathscr{I}(C(\langle x, X\rangle))$, and $\langle x, y\rangle \leqslant T \in \mathrm{Syl}_2(C_G(X))$, $C_T(y) = C_T(Y)\langle x\rangle$. By 3.3, $C_T(Y)$ is cyclic, so if $C_T(y) \neq \langle x, y \rangle$, then $\langle y \rangle$ char $C_T(Y)$, so $T = C_T(y)$ is of 2-rank 2. If $C_T(y) = \langle x, y \rangle$, the same remark holds by a lemma of Suzuki. Thus $m(T) = 2$. But $\langle x, T \cap X_2 \rangle$ is of rank 3, a contradiction.

(6.4) *Let* $X \in \mathscr{C}(\mathscr{I}(G)) \cap \mathscr{F}^*$, *and* $T \in \mathrm{Syl}_2(C_G(X))$. *Assume Hypothesis* 6.2. *Then one of the following holds*:

(1) *T is cyclic.*

(2) $X \cong L_3^{\varepsilon}(3)$, $L_2(9)$, *or* $\Omega_5(3)$ *and* $m(T) = 2$.

(3) $X \cong U_3(3)$ *or* $L_2(9)$ *and T has sectional* 2-*rank* 3.

Proof. By 6.3 and Hypothesis A, we may take $X \leqslant L \in \mathscr{C}^*$; then without $C_T(t) = R \in \mathrm{Syl}_2(C_G(X\langle t \rangle))$. Set $U = C_R(L)$. As $L \in \mathscr{C}^*$, U is cyclic. As $[R, X] = 1$, we conclude from Section 2 and Hypothesis 6.2 that one of the following holds:

(i) $|R : U| \leqslant 2$.

(ii) $X \cong L_2(9)$, $L/O(L) \cong \Omega_6^-(3)$, and $R/U \cong E_4$.

(iii) $X \cong U_3(3)$, $L/O(L) \cong \Omega_6^-(3)$, $R/U \cong Z_4$.

Notice that if $R = T$, then with 3.3, the lemma holds. So assume $R \neq T$. Suppose next that $U \neq \langle t \rangle$. Then by 1.1. $\langle U, U^s \rangle = U \times U^s \leqslant R$ for $s \in T - R$. Hence (iii) holds and $R = \langle U^T \rangle \cong Z_4 \times Z_4$. Indeed $T \cong Z_4$ wreath Z_2, so (2) holds.

So take $U\langle t \rangle$. If $|R : U| \leqslant 2$ then by a lemma of Suzuki, (2) holds. Thus we may take $|R : U| = 4$ and (ii) or (iii) holds. Hence $|R| = 8$. We may assume U is not characteristic in R, so $R \cong E_8$ or $Z_4 \times Z_2$ in (ii) and (iii), respectively.

Suppose $R \cong E_8$. Then there is $r \in R$ such that $J = E(C_L(r)) \cong \Omega_5(3)$. By Hypothesis A and 6.3, $J \leqslant I \in \mathscr{C}(C_G(r)) \cap \mathscr{F}$. With Hypothesis 6.2, either $J = I$ or $I/O(I) \cong \Omega_6^-(3)$. We may assume (3) does not hold, so $T \neq R$. Thus there is $s \in N_T(R) - R$ with $s^2 \in R$. Then s centralizes a member of $\langle r, t \rangle - U$, which we may take to be r. If $I \in \mathscr{C}^*$ we are done by induction on $|R|$. So take $I = J$. Then $C_T(r) \cong Z_2 \times C_T(I\langle r \rangle)$ with $\langle r, t \rangle = C_T(I\langle r, t \rangle)$, so that $C_T(I\langle r \rangle)$ is dihedral or semidihedral. As $|C_T(r)| > 8$, $\langle r \rangle$ char $C_T(r)$, so $T = C_T(r)$ and (3) holds.

This leaves $R \cong Z_4 \times Z_2$. Here 1.2 implies that (3) holds.

(6.5) *Let* $X \in \mathscr{C}(\mathscr{I}(G)) \cap \mathscr{F}^*$ *and assue Hypothesis* 6.2. *Then* $m(N_G(X)) \leqslant 6$.

Proof. This is a consequence of 6.3 and the structure of $\mathrm{Aut}(X)$.

(6.6) *Assume* $L/O(L) \cong \Omega_6^-(3)$ *and* $z \in \mathscr{I}(L)$. *Then* $tz \in t^G$.

Proof. By the Z^*-theorem there is $g \in G - N(L)$ with $s = t^g \in C(t)$. If s induces an inner automorphism on L then by 3.4.3, $s \notin L$, so $s \in (tz)^L$ and the lemma holds. So assume s induces an outer automorphism on L. Then by 2.10 either $X = E(C_L(s)) \neq 1$ or s is of type $i(3, \alpha)$, in which case $C_L(s)$ has a normal subgroup $X = X_1 \times X_2$ with $X_i \cong A_4$. As $C_G(L)$ has cyclic Sylow 2-groups, $O^{2'}(X) \leqslant L^g$ by Theorem 2 in [2]. But by 2.16, $su \in s^L$ for some $u \in \mathscr{I}(X)$, so $tz \in (su)^G = t^G$.

(6.7) *Assume* $L/O(L) \cong \Omega_6^-(3)$ *and Hypothesis* 6.2 *holds. Let* $E_{2^8} \cong E \leqslant G$. *Then*

(1) *If* $x \in \mathscr{I}(G)$ *with* $m(C_E(x)) \geqslant 6$ *then* $C_G(x)$ *has no component in* $\mathscr{F}^*$.

(2) *Assume* $z \in \mathscr{I}(L)$ *and each involution in* $N(L)$ *inducing an automorphism of type* $i(3, \alpha)$ *on* L *is in* t^G.

Then

(i) $m(C_E(y)) < 6$ *for each* $y \in \mathscr{I}(N(L)) - z^L$.

(ii) $C_E(s)^\# \subseteq z^G$ *for each* $s \in t^G$.

Proof. Assume the hypothesis of (1) with $D = C_E(x)$ and $\mathscr{C}(C_G(x)) \cap \mathscr{F}^*$ nonempty. By 6.5, $m(C_G(x)) \leqslant 6 \leqslant m(D)$, so $x \in D$. But then $m(C_G(x)) \geqslant m(E) > 6$, a contradiction.

Next assume the hypothesis of (2). Then $E(C_L(y)) \neq 1$, so by Hypothesis A $\mathscr{C}(C_G(y)) \cap \mathscr{F}^*$ is nonempty, so by (1), $m(C_E(y)) < 6$. In applying this to $y \in C_E(t)$, we obtain (2)(ii).

(6.8) *Assume* $L/O(L) \cong \Omega_6^-(3)$, $E_{16} \cong B \leqslant L$, $t \neq t^g \in tB$, *and assume* $B \leqslant L^g$. *Then* $\langle t \rangle \in \mathrm{Syl}_2(C_G(L))$ *and* $N_G(\langle t, B \rangle)$ *is 2-transitive on* tB.

Proof. Let $A = \langle t, B \rangle \leqslant S \in \mathrm{Syl}_2(N_G(L))$ and assume $X = C_S(L) \neq \langle t \rangle$. Let $Y = C_G(\langle t, L \rangle)$. $\mathrm{Aut}_L(B) \cong A_6$ is transitive on $Bt - \{t\}$, so $H = \langle N_L(B), N_{L^g}(B) \rangle$ is 2-transitive on Bt. Further $Y \leqslant C(t^g) \leqslant N(Y^g)$, so $[Y, Y^g] \leqslant Y \cap Y^g \leqslant O(Y)$. Then H acts on $K = \langle Y^h : t^h \in tB \rangle$ and as H is 2-transitive on tB, $K/O(K)$ is abelian. In particular $U = S \cap K$ is abelian with $A \leqslant U$, so $U = \langle X^{N_H(U)} \rangle \leqslant C(B)$. But a maximal 2-signalizer for $N_{LS}(A)/X$ is isomorphic to E_{16} or E_{32} by 2.16, so $A \leqslant \Phi(U) \leqslant X$, a contradiction.

7

In this section assume $L \cong \Omega_6^+(3)$. Collecting results in previous sections, we have

(7.1) *$L \in \mathscr{C}^*$.*

(7.2) *$C(L)$ has a cyclic Sylow 2-groups.*

(7.3) *If $z^g \in C(t)$ then z^g induces an automorphism of type $i(3, \alpha)$ or $i(4, +)$ on L.*

Proof. This is a consequence of 2.11 and 3.7.

(7.4) *Either*

(1) *$tz \in t^G$, or*

(2) *each member of $t^G \cap C(t) - \{t\}$ is of type $i(1, \pm 1)$ on L.*

Proof. Assume otherwise. By the Z^*-theorem there is $s = t^g \in C(t) - \langle t \rangle$. Set $X = E(C_L(s))$ unless s is of type $i(3, \alpha)$ on L, where $X = O_2(O^2(C_L(s)))$. t is not fused to z by 7.3 and we are assuming neither (1) or (2) holds, so by 2.17.1 either we may take $z \in X$ and $sz \in s^L$, or s is of type $i(4, -)$ on L, and $u \in \mathscr{I}(X)$ is of type $i(4, -)$. However, by Theorem 2 in [2], $X \leqslant L^g$, so in the first case $z \in (z^g)^{L^g}$ by 7.3, and as $sz \in z^L$, $tz \in t^G$, a contradiction. So the second case holds. Then $s = v$ or tv, $v \in \mathscr{I}(L)$ of type $i(4, -)$. Notice $uv \in z^L$. By 7.3, $u \in (u^g)^{L^g}$. Thus if $s = v$ then uv is of type $i(4, -)$ on L^g, contradicting $uv \in z^L$ and 7.3. So $s = tv$. $X = E(C_{L^g}(t)) = E(C_L(s)) \cong L_2(9)$ contains u of type $i(4, -)$ in L^g, so t is of type $i(4, -)$ on L^g by 2.17.3. By symmetry $v = st \in L^g$, and $uv \in (z^g)^{L^g}$. But $suv = tu \in (tv)^L = s^L$, a contradiction.

(7.5) (1) *$S \cap L$ has two normal E_{16}-subgroups B_1 and B_2.*

(2) *$N_L(B_i)/B_i \cong S_5$ acts naturally on B_i, so $B_i = \langle z^{N_L(B_i)} \rangle$.*

(3) *Either $C_G(\langle t, B_i \rangle) = C_G(\langle t, L \rangle) B_i$ or some $s_i \in S$ induces an automorphism of type $i(1, \alpha_i)$ on L, $C_G(\langle t, B_i \rangle) = C_G(\langle t, L \rangle B_i \langle s_i \rangle)$, $[s_i, N_L(B_i)] = 1$, and $[s_i z, N_L(B_{3-i})] = 1$ with $s_i z$ of type $i(1, -\alpha_i)$ on L.*

Proof. See 2.17.

(7.6) *$\langle t \rangle = C_S(L)$.*

Proof. Let $s = t^g \in S - \langle t \rangle$ and $C(s)\alpha = C(s)/C(L^g)$. Assume x is of order 4 in $C_S(L)$. Suppose $s = tz$. Then $x \in C(s)$ and t is of type $i(4, +)$ on L^g. This is impossssible by 2.11, since $Z_4 \cong \langle x\alpha \rangle \trianglelefteq C_{C(s)\alpha}(t\alpha)$.

So by 7.4, t is of type $i(1, \beta)$ on L^g. By [5] we may choose s so that $x \in C(s)$. As $Z_4 \cong \langle x\alpha \rangle \leqslant C(E(t\alpha)))$, 2.11 again supplies a contradiction.

If $tz \in t^G$ choose $s = t^g = tz$ and let $B = B_1$ or B_2 be chosen as in 7.5. Otherwise by 7.4 we may choose $s = t^g \in C_S(S \cap L)$ of type $i(1, \alpha)$ on L and $B = B_1$ or B_2 as in 7.5 with $[s, N_L(B)] \neq 1$. In either case let $A = \Omega_1(C_S(B))$, $M = N_G(A)$, and $M^* = M/C(A)$.

(7.7) (1) *$A \cap L^g \cong E_{16}$ and $(L^g \cap M)/(L^g \cap A) \cong S_5$.*
(2) $[t, L^g \cap M] \neq 1$.

Proof. Let $X = O^2(C_{M \cap L}(s))$ and $Q = O_2(X)$. $X \cong Z_3/Q_8 * Q_8$. As $\mathrm{Out}(L) \cong E_4$, $X \leqslant L^g C(L^g)$ and then $Q \leqslant L^g$ by 7.2. Let $T = \langle t, s, z^G \cap S, Q \rangle$. $S \cap L = \langle Q, z^G \cap S \cap L \rangle$, so by 7.3 and the choice of s, either $T = \langle t, S \cap L \rangle$ or T is the subgroup of S inducing automorphisms in $PO_6^+(3)$ on L. By symmetry between s and t, $T \in (T^g)^{L^g}$. Thus $D_i = \Omega_1(C_S(B_i)) \cap L^g$, $i = 1, 2$, are the normal E_{16}-subgroups of $T \cap L^g \in \mathrm{Syl}_2(L^g)$, so that (1) holds. As $C_M(\langle t, s \rangle)$ is solvable, (2) holds.

(7.3) *$A = \langle t, B \rangle$.*

Proof. If not $A = \langle r, t, B \rangle \cong E_{64}$ with $\langle r, t \rangle = C_A(L \cap M)$ and r or type $i(1, \alpha)$ on L. Let $X = O^2(C_M(\langle t, s \rangle))$. Then $\langle r, s, z \rangle = C_A(X)$ and $\langle r, t \rangle$ and $C_A(L^g \cap M)$ are hyperplanes of $C_A(X)$, so we may choose $r \in C_A(L^g \cap M)$. $\Omega_5(3) \cong E(C_L(r)) \leqslant K \in \mathscr{C}(C_G(r)) \cap \mathscr{F}$ by Hypothesis A and 3.6. Thus $K \cong \Omega_5(3)$ or $\Omega_6^+(3)$ by 6.1, so $L \cap M = C_M(r)^\infty = L \cap M^g$, contradicting 7.7.2.

(7.9) (1) *$\mathrm{Aut}_G(L) \cap PO_6^+(3) = \mathrm{Inn}(L)$.*
(2) *$AO(C(t)) = AO(M) = C_G(A)$.*

Proof. Part (1) is equivalent to (2); if either fails then by 7.5 there is $x \in C_S(A)$ inducing an automorphism of type $i(1, \alpha)$ on L. By 7.8, $x^2 = t$, so $\langle t \rangle = \Phi(C_S(A))$ and hence $M \leqslant C(t)$ by a Frattini argument. This contradicts 7.7.2.

(7.10) (1) *$s = tz \in t^G$.*
(2) *$z^G \cap C(t) = z^L$.*
(3) *$B = A \cap L = \langle z^G \cap A \rangle \trianglelefteq M$.*
(4) *$M^* = O_2(M^*)(L \cap M)^*$ with $O_2(M^*) = C_{M^*}(B) \cong E_{16}$ regular on $tB = t^G \cap A$.*

Proof. Part (1) is a consequence of 7.9.1 and the choice of s. Now (1), 7.3, and 7.9 imply (2), and (2) implies (3). Next $B^\# = \Delta \cup u^{(M \cap L)}$, where $\Delta = z^{(M \cap L)}$ is of order 5. By (2), $z^G \cap A = \Delta$. So as $(M \cap L)^* \cong S_5$, $M^* = K^*(M \cap L)^*$, where $K^* = M^*_\Delta$. Then K_Δ centralizes $\langle \Delta \rangle = B$ and A/B, so $K^* = O_2(M^*)$ and (4) holds.

Let E be a Sylow 2-group of the preimage of $O_0(M^*)$, so that $E/A \cong E_{16}$ and $M^* = N_M(E)^*$. The map $eA \to [e, t]$ is an $(L \cap M)$ – isomorphism of E/A and B, so E/A is the natural module for $(L \cap M)^* \cong S_5$. Then $E/B \cong E_{32}$ or $D_8 * Q_8$. In the latter case $e^2 \in tB$ for some $e \in E$, so

$e^2 \in C_A(e) = B$, a contradiction. As E/A is the natural module for $(L \cap M)^* \cong S_5$, $[E, M \cap L] = U$ is a complement to $\langle t \rangle$ in E. By [12], $U \cong (Z_4)^4$ or E_{2^8}.

Let $b \in B^{\#}$ be of type $i(2, -)$. By 7.9 and 6.3,

$$E(C_L(b)) = X \leqslant Y \in \mathscr{C}(C_G(b)) \cap \mathscr{F}$$

with $Y \trianglelefteq C_G(b)$. $X \cong L_2(9)$ so $Y = X$ or $Y \cong \Omega_5(3)$, $\Omega_6^{\varepsilon}(3)$, or $L_2(81)$, and $Y \in \mathscr{C}^*$. As $Y \trianglelefteq C_G(b) \geqslant E$, E acts on $C_A(Y)$. $C_A(X) = \langle t, b \rangle$, so $C_A(Y) = \langle b \rangle$ or $\langle t, b \rangle$. As $[t, E] = B$, it is the former. Hence $Y \neq X$, so $Y \in \mathscr{C}^*$, and $C_E(Y)$ is cyclic by 3.3. Thus $\mathrm{Aut}_U(Y)$ is an abelian subgroup of $\mathrm{Aut}_G(Y)$ of order at least 64. As $Y \cong \Omega_5(3)$, $\Omega_6^{\varepsilon}(3)$, or $L_2(81)$, this is impossible.

This contradiction establishes the following result:

THEOREM 7.11. *If G satisfies Hypothesis A, $F^*(G) \notin \mathrm{Chev}(3)$, and $L \in \mathscr{C}(\mathscr{T}(G))$, then $L/O(L)$ is not isomorphic to $\Omega_6^+(3)$.*

8

In this section we assume $L \cong \Omega_6^-(3)$. Collecting results from previous sections we have

(8.1) $L \in \mathscr{C}^*$.

(8.2) *$C_G(L)$ has cyclic Sylow 2-groups.*

(8.3 $tz \in t^G$.

Proof. See 6.5.

Throughout this section and in Section 9 let $E_{16} \cong B \leqslant L \cap S$. Let $g \in G$ with $t^g = tz$. By 2.16, $N_L(B)/B \cong A_6$ acts naturally on B. In this section we prove

THEOREM 8.4. *Some involution in $C(t)$ induces an automorphism of type $i(3, a)$ on L.*

Hence in the remainder of this section assume G is a counter example of Theorem 8.4. In particular by 3.7, 2.10, and 8.3.

(8.4) $z^G \cap C(t) = z^L$.

(8.5) $S \cap L = S \cap L^g$.

Proof. $S \cap L = \langle z^L \cap S \rangle = \langle z^G \cap S \rangle$ by 8.4. Further $S \leqslant C(tz) \leqslant N(L^g)$, so $S \cap L = S \cap L^g$.

(8.6) $C_S(L) = \langle t \rangle$.

Proof. See 8.5 and 6.17.

(8.7) *No element of $N(L)$ induces an automorphism of $i(1\ \alpha)$ or $i(3, \alpha)$ on L.*

Proof. If so, from Section 2 there is $x \in C_S(B)$ of type $i(3, \alpha)$ on L. As Theorem 8.4 fails, $x^2 = t$. Let $X = \langle x, B \rangle$. Then $X = XL^g \cap C(B) \trianglelefteq N_{L^g}(B)$, so $N_{L^g}(B)$ acts on $\Phi(X) = \langle t \rangle$, impossible as $\langle tz \rangle = C_{\langle t,B \rangle}(N_{L^g}(B))$.

Set $A = \langle t, B \rangle$ and $N = N_G(A)$. By 8.7 and Section 2,

$$AO(N) = C_{N(L)}(A) = C_G(A).$$

By 6.8, N is 2-transitive on $tB = t^G \cap A$. Hence we may repeat, essentially verbatim, Lemmas 4.6 through 4.13 of Finkelstein [6], omitting Lemma 4.8 and the second sentence of the proof of Lemma 4.9, which are extraneous. In particular these lemmas show $N = EC_N(t)$, where $E \cong E_{2^8}$, $B = C_E(t) = [E, t]$, and $EO(N) \trianglelefteq N$. Set $X = N_G(E)$ and $X^* = X/C_G(E)$. By 4.12 in [6], $C_G(E) = E \times O(X)$, while by 4.13 in [6], $C_{X^*}(t^*) = C_X(t)^*$.

$$C_X(t)^*/\langle t^* \rangle \cong \mathrm{Aut}_{C(t)}(B) \cong A_6 \text{ or } S_6$$

by 2.16. By 6.7, 7.11, and 8.7

(8.8) (1) $m(C_E(y)) < 6$ *for each* $y \in \mathscr{I}(C(t)) - z^L$.
(2) *If* $s \in t^G$ *then* $C_E(s)^\# \subseteq z^G$.

(8.9) $\mathrm{Aut}_{C(t)}(B) \cong A_6$.

Proof. Assume not, so that $\mathrm{Aut}_{C(t)}(B) \cong S_6$. Then there is $x \in C_N(t) - L$ inducing a transvection on A with $x^2 \in \langle t \rangle$. We show x is an involution and $m(C_E(y)) \geqslant 6$ for $y = x$ or xt, contradicting 8.8.1.

The map $eB \to [e, t]$ is a $C_X(t)$-isomorphism of E/B with B, so y induces a transvection on E/B. As B centralizes y^2, $C_{E/B}(y) \leqslant C_E(y^2)/B$, so $y^2 \neq t$ and hence x and y are involutions. Assume $m(C_E(y)) < 6$, and let $fB = [E/B, y]$. We may choose $f \in C_E(y)$. Next replacing x by a suitable member of Bx, we may assume x centralizes a subgroup W of order 3 in $L \cap N$. $C_{E/B}(y) = [E/B, W] \oplus \langle fB \rangle$ and $[E, W] \cong E_{16}$, so either $[E, W, y] = 0$ or $C_{[E,W]}(y) = [B, W]$. In the first case $C_{E/B}(y) = C_E(y)\,B/B$, so $m(C_E(y)) \geqslant 6$, a contradiction. Thus $C_E(y)\,B/B = [E/B, y]$.

Let $Q=O_2(C_L(z))$. Then $C_E(Q)=\langle e,z\rangle\cong E_4$ and we may choose $x\in S$ so that $eB\neq fB$. But $\langle t,x\rangle$ acts on $\langle e,z\rangle$ with $[e,t]=z$, so x or xt centralizes e, contradicting $eB\notin[E/B,y]=C_E(y)B/B$.

(8.10) $X^*=(X_1^*\times X_2^*)\langle t^*\rangle$ *with* $(X_1^*)^t=X_2^*\cong A_6$.

Proof. This is established in 4.14 of [6]. Lemma 8.8.2 takes the place of the assertion in the last paragraph of that proof that $C_G(t)$ has two G-classes of involutions.

Set $\bar{X}=X/O(X)$. By 4.15 in [6]:

(8.11) $\bar{X}=(\bar{X}_1\times\bar{X}_2)\langle\bar{t}\rangle$ *with* $\bar{X}_1^t=\bar{X}_2\cong A_6/E_{16}$.

From 8.6, 8.7, and Section 2, either $S=\langle t\rangle\times(S\cap L)$, or $S=\langle t,\gamma,S\cap L\rangle$ where $\gamma\in S$ induces an involutory automorphism of L not in $PO_6^-(3)$. The former is impossible by Theorem B in [6], so the latter holds. Then from 2.10, $E(C_L(\gamma))=K_o\cong L_2(9)$. By Hypothesis A and 6.3, $K_0\leqslant\mathscr{C}(C_G(\gamma))$ and $K\in\mathscr{F}$.

Let $S\cap X\leqslant T\in\mathrm{Syl}_2(X)$. Then $J(T)=U_1\times U_2$, where $U_i=J(T\cap X_i)$ is isomorphic to a Sylow 2-group of $L_3(4)$. Let Y_i be of order 3 in $N_{X_i}(U_i)$, t_i an involution in $T\cap X_i$ inverting Y_i and $T_i=\langle t_i,U_i\rangle=T\cap X_i$. By 4.8 in [6], $\langle t,E\rangle$ is Sylow in $C_G(B)$. Moreover $BB^\gamma=J(S\cap L)$, so

$$J(C_T(B^\gamma))=E_0\cong E_{2^8}$$

and in particular $E_0\in E^G$. But E and E_0 are the unique members of $\mathcal{O}\!\mathcal{l}(T)$ normal in T, so if $T\in\mathrm{Syl}_2(G)$. Then $E_0\in E^{N(T)}$ and $N(T)$ acts on $\{E,E_0\}$, so $|N(T)\colon N(E)|=2$, contradicting $T\in\mathrm{Syl}_2(G)$.

So pick α to be a 2-element in $N(T)-T$. $m(C_X(s))=8$ for each $s\in\mathscr{I}(T_1T_2)$, so $t^G\cap T_1T_2$ is empty. On the other hand T is transitive on the involutions in $T-T_1T_2$, so $T_1T_2\trianglelefteq N(T)$ and $N(T)=T(N(T)\cap C(t))$. Thus we may take $\alpha=\gamma$ and $R=(R\cap N)\langle\gamma\rangle\leqslant N(T)$.

Next $Y=Y_1Y_2O(X)$ is a Hall group of the pointwise stabilizer of $\mathcal{O}\!\mathcal{l}(T)$, so γ acts on YT. By the Krull–Schmidt theorem, γ permutes $\{Y_1T_1O(X),Y_2T_2O(X)\}$ and replacing γ by γt if necessary, we may assume γ acts on $Y_iT_iO(X)$. In particular as $\gamma^2\in\langle t\rangle$ but t does not act on $Y_iT_iO(X)$, γ is an involution.

The map $\bar{x}\to\overline{xx^t}$ is a γ-isomorphism of $\bar{Y}_i\bar{T}_i$ with $\overline{YT_1T_2}\cap C(t)=\overline{L\cap YT_1T_2}$, so $C_{\bar{Y}_i\bar{T}_i}(\gamma)\cong C_{L\cap YT}(\gamma)$, and we may choose γ so that $C_{L\cap YT}(\gamma)\cong S_4$. Recall $K_0=E(C_L(\gamma))\leqslant K\in\mathscr{C}(C_G(\gamma))\cap\mathscr{F}$. By 6.3, $K\trianglelefteq C_G(\gamma)$. Now

$$S_4\cong K_1=L\cap YT\leqslant K_0\leqslant K.$$

But $C_{U_1U_2}(\gamma)=U=[U,K_1]\cong E_{16}$, so U is a t-invariant E_{16}-subgroup of K.

Then $m(K) = 4$, so $K/O(K) \cong \Omega_6^-(3)$ or $\Omega_5(3)$. As $C_G(\langle \gamma, t\rangle)/O(C_G(\langle \gamma, t\rangle)) \cong E_4 \times M_{10}$ by Section 2, $K/O(K) \cong \Omega_6^-(3)$ and t induces an automorphism on K not in $PO_6^-(3)$. But then t interchanges the two K-classes of E_{16}-subgroups of K, contradicting $t \in N(U)$.

This contradiction completes the proof of Theorem 8.4.

9

In this section we continue the hypothesis and notation of the previous section. By Theorem 8.4 there is $t_0 \in \mathscr{I}(C(t))$ inducing an automorphism of type $i(3, \alpha)$ on L, and by 2.16 we may take $[t_0, B] = 1$. Set $A = \langle t, t_0, B\rangle$.

(9.1) *For some choice of* $E_{16} \cong B \leqslant L \cap S$, $B = A \cap L^g$.

Proof. $B^\# = z^{N_L(B)}$, so if $B \neq A \cap {}^g$ there is $z' \in z^{N_L(B)} - L^g$. By 3.7 and 2.10, z' is of type $i(3, \beta)$ on L^g. $A \cap O^2(C_L(z)) \leqslant L^g$, so $z' \notin O^2(C_L(z))$, and hence z' interchanges the two subnormal $SL_2(3)$-subgroups of $C_{L^g}(z)$. Hence by 2.16.6, zz' is of type $i(1, -\beta)$ on L^g. Notice $tz' \in (tz)^L \in t^G$ and $tz' = tz(zz')$ is of type $i(1, -\beta)$ on L^g. So there is $s = t^h$ of type $i(1, \gamma)$ on L. By 2.16 we may choose s to centralize $E_{16} \cong B' \leqslant S \cap L$. Set $A' = \langle t, s, B'\rangle$. $B' \leqslant E(C_L(s)) = E(C_{L^h}(t))$, so $B' = A \cap L^h$. By 6.8, $N(\langle t, B'\rangle)$ is 2-transitive on tB', so as $z \in B'$, $B' = A' \cap L^g$, and the lemma holds.

(9.2) $\langle t\rangle = C_S(L)$.

Proof. See 9.1 and 6.8.

In the remainder of this section choose B so that $B = A \cap L^g$. Set $N = N_G(B) \cap C(A/B)$. Then $N_0 = \langle N_{L^g}(A), N_L(A)\rangle \leqslant N$. By 9.2 and 2.16, A is Sylow in $C_G(A)$ and $C_G(\langle t, B\rangle)$, so $C_N(A) = C_N(\langle t, B\rangle) = O(N)A$. Set $\bar{N} = N/C_N(A)$. By 6.8, N is 2-transitive on tB, and then appealing to 4.6 and 4.7 in [6], $\bar{N} \cong \mathrm{Aut}_{N(L)}(B)/E_{16} \cong A_6/E_{16}$ or S_6/E_{16}. Let E_1 be a Sylow 2-group of the preimage of $O_2(\bar{N})$.

(9.3) $B = C_A(E_1)$.

Proof. Assume $s \in C_A(E_1) - B$. From Section 2, we may assume s induces an automorphism of type $i(1, \alpha)$ on L and $C_{L \cap N}(s) \cong A_5/E_{16}$. By 6.3 and Hypothesis A there is a normal component K of $C_G(s)$ in $\mathscr{F}$ containing $E(C_L(s)) \cong \Omega_5(3)$. Thus $K/O(K) \cong \Omega_5(3)$ or $\Omega_6^-(3)$. But $E_0 = [E_1, C_{L \cap N}(s)] \leqslant K$, whereas $|E_0| \geqslant 2^8$ and $|K_2 \leqslant 2^7$, a contradiction.

(9.4) *Let* $s \in A - B$. *Then there is* $L_s \in \mathscr{C}(C_G(s))$ *with* $L_s/O(L_s) \cong$

$\Omega_6^-(3)$, $N \cap L_s \cong A_6/E_{16}$, *and* $B = O_2(N \cap L_2)$. *In particular we have symmetry between s and t.*

Proof. Because of 9.3, E_1 is transitive on sB, so by a Frattini argument $C_N(s)$ covers $\bar{N}/O_2(\bar{N})$, and s is fused to an element inducing an automorphism of type $i(1, \alpha)$ on L. This forces the existence of L_s. $B = [B, C_N(s)^\infty] \leqslant L_s$, so $B = O_2(N \cap L_s)$.

(9.5) $N = EC_N(t)$, *where* $E_{2^8} \cong E$, $EO(N) \trianglelefteq N$, and $B = C_E(t) = [E, t]$.

Proof. Set $N\alpha = N/BO(N)$. Then $(E_1)\alpha \cong E_{64}$ with $A\alpha \leqslant Z(N\alpha)$ and $N/E_1O(N) \cong A_6$ or S_6 acting naturally on E_1/A. In particular from the cohomology of that representation either $(E_1)\alpha = (A\alpha) \oplus (E\alpha)$ for some $E \leqslant E_1$, or $(E_1)\alpha = (A_1)\alpha \oplus (E\alpha)$ with $E\alpha \cong E_{32}$. In the first case Lemma 4.11 in [6] implies $E \cong E_{2^8}$. Evidently $B = C_E(t) = [E, t]$, so E is transitive on the involutions in tE, and hence $N = EC_N(t)$ by Frattini argument. In the second case by 9.4 we may take $t \in E$. Then Lemmas 4.9 and 4.10 in [6], modified as indicated in Section 8, supply a contradiction. Thus the lemma holds.

We next observe, as in Lemma 4.12 of [6], that $E_1 \in \mathrm{Syl}_2(C_G(B))$ and $C_G(E) = E \times O(C_G(E))$. For if x is a 2-element in $C_G(B) \cap N(E_1)$ then x acts on $E = J(E_1)$ and fixes some coset sE of E_1/E. By 9.4 we may take $t \in sE$ and by a Frattini argument take $x \in C_{E_1\langle x\rangle}(t) = A \leqslant E_1$.

Next by 9.4 and 6.6,

(9.6) (1) $m(C_E(y)) < 6$ *for each* $y \in \mathscr{I}(N(L)) - z^L$.

(2) *If* $s \in t^G$ *then* $C_E(s)^{\#} \subseteq z^G$.

Now repeat the proof of 8.9 verbatim to obtain

(9.7) $\mathrm{Aut}_{N(L)}(B) \cong A_6$.

Then let $X = N_G(E)$ and $X^* = X/C(E)$. Lemma 9.7 and a Frattini argument show $C_{X^*}(s^*) = C_X(s)^* \cong E_4 \times A_6$ for each $s^* \in (A^*)^{\#}$. Then by [4], $X^* = N_{X^*}((X \cap L)^*)$. In particular taking $T \in \mathrm{Syl}_2(X)$, we have $E = J(T)$ and then $T \in \mathrm{Syl}_2(G)$.

Let u be an involution in $(L \cap X) - B$, so that $tu \in t^G$. Then $C_E(t) \nleqslant B$, so by 9.6.2, $z^G \cap E \nsubseteq B$. This is impossible as $E = J(T)$, so that X controls fusion in E, whereas $B \trianglelefteq X$.

This contradiction establishes

THEOREM 9.8. *If* G *satisfies Hypothesis* A, $F^*(G) \notin \mathrm{Chev}(3)$, *and* $L \in \mathscr{C}(\mathscr{I}(G))$, *then* $L/O(L)$ *is not* $\Omega_6^-(3)$.

10

In this section we assume $L \cong \Omega_5(3)$.
Once again we have

(10.1) $L \in \mathscr{C}^*$.

(10.2) $C_G(L)$ *has cyclic Sylow* 2-*groups.*

(10.3) $\langle z, t\rangle = \Omega_1(Z(S))$.

In the remainder of the paper let $H = C_G(z)$ and $H/\langle z\rangle = H\alpha$. Let $K_i \trianglelefteq\trianglelefteq C_L(z)$, $i = 1, 2$, with $z \in K_i \cong SL_2(3)$. Set $K = K_1K_2$, $Q_i = O_2(K_i)$, and $Q = O_2(K)$. Set $K_0 = O(C(t))\,K$. *Then* $K_0 = O^2(C_G(\langle z, t\rangle) = O^2(C_H(t))$.

(10.4) *If* $t\alpha$ *is strongly closed in* $\langle Q\alpha, t\alpha\rangle$ *with respect to* H, *then* $t^H \cap C(t\alpha) \subseteq N(K_1 O(K_0))$.

Proof. Assume otherwise and let $W = \langle Q, t\rangle$, and $h \in H$ with $s = t^h \in C(t\alpha)$ and $K_1^s O(K_0) = K_2 O(K_0)$. Then $C_{K_0}(s) = C_{O(K_0)}(s) \times \langle z\rangle \times X$ with $X \cong A_4$. $X \leqslant O^2(C_H(s)) = K_0^h$, so $O_2(X) \leqslant Q^h$. Thus as $t\alpha$ is strongly closed in $W\alpha$, and as $C_Q(s) = [W, s] \leqslant W^h$, $s^H \cap s[W, s] \subseteq \{s, sz\}$. Thus $|Q{:}\, C_Q(s)| \leqslant 2$, which is not the case.

(10.5) $zt \in z^G$.

Proof. Assume not. By 3.7, $t \notin z^G$, so by 10.3, t is weakly closed in $Z(S)$ and hence $S \in \mathrm{Syl}_2(G)$. Set $W = \langle Q, t\rangle$. Then $E_{32} \cong W\alpha$ and from the structure of $\mathrm{Aut}(\Omega_5(3)) \cong O_5(3)$ we obtain:

(10.5.1) *If* $1 \neq E_{2^n} \cong U/W \leqslant S/W$ *then* $n \leqslant 2$ *and* $m(W\alpha/C_{W\alpha}(U)) > n$.

We conclude from 10.5.1 that $W\alpha = J(S\alpha)$, so that $W\alpha$ is weakly closed in $S\alpha$. $\langle t\alpha\rangle = Z(W)\,\alpha \trianglelefteq N_{H\alpha}(W\alpha)$, and as $W\alpha$ is weakly closed and abelian, $N_{H\alpha}(W\alpha)$ controls fusion in $W\alpha$, so $t\alpha$ is strongly closed in $W\alpha$. Indeed t is strongly closed in S with respect to H. For if $h \in H - C(t)$ with $t^h = s \in S$, then by 10.4, $s \in N(K_1)$. t is weakly closed in W and $\Omega_1(C_S(K_1)) \leqslant W$, so s is faithful on K_1. Hence $sz \in s^{K_1}$, contradicting $tz \notin t^G$.

So $t \in Z^*(H)$ by the Z^*-theorem. Thus $KO(H) \trianglelefteq H$. But now Corollary III in [3] contradicts $F^*(G) \notin \mathrm{Chev}(3)$.

In the remainder of the paper let $g \in G$ with $t^g = tz$.

(10.6) $\langle t\rangle = C_S(L)$.

Proof. Let $C(tz)\,\beta = C(tz)/C(L^g)$. Assume x is of order 4 in $C_S(L)$. Then

$x \in C(tz)$ and t is of type $i(4,+)$ on L^g. This is impossible by 2.6, since $Z_4 \cong \langle x\beta \rangle \trianglelefteq C(t\beta)$.

By 2.18, $J(S \cap L) = B \cong E_{16}$. Set $A = \langle B, t \rangle$, $M = N_G(A)$, and $M^* = M/C(A)$.

(10.7) (1) *B is the natural module for $(M \cap L)^* \cong A_5$.*

(2) *$C_M(t)^* \cong A_5$ or S_5.*

(3) *Either*

(i) *$A = J(S)$, or*

(ii) *there is $r \in \mathcal{I}(S)$ of type $i(4,-)$ on L and A and $A_2 = \langle C_Q(r), r, t \rangle$ are the normal E_{32}-subgroups of S.*

Proof. This follows from 2.18.

(10.8) *$t^G \cap L$ is empty.*

Proof. $S/(S \cap L)$ is abelian as $\mathrm{Out}(L)$ is of order 2 and $\langle t \rangle = C_S(L)$. Thus $t \notin [S,S]$. But if $x \in \mathcal{I}(L)$ and $C_S(x) \in \mathrm{Syl}_2(C(\langle t, x \rangle))$, then $x \in [C_S(s), C_S(x)]$.

(10.9) *$A \trianglelefteq N_G(S)$.*

Proof. Assume otherwise. Then by 10.7.3, S has two normal E_{32}-subgroups $A = A_1$ and A_2 which are interchanged by $g_0 \in N_G(S)$. g_0 acts on $\Omega_1(Z(S)) = \langle z, t \rangle$ interchanging t an tz, so without loss $g = g_0$ is a 2-element. Let $T = \langle g, S \rangle$. Then $S = C_T(t)$ is of index 2 in T. Notice $A_i = \langle a_i, t, A_i \cap Q \rangle$ with $K_1^{a_i} = K_2$. Hence as $A_1^g = A_2$, g induces an outer automorphism on $K_0/O(K_0)$ and $K_0 T/O(K_0)\langle t, z \rangle \cong S_4$ wreath Z_2. Thus we may choose $[g, K_2] \leqslant O(K_0)$, $g^2 \in C(K_0)$, and g to induce an outer automorphism on $K_1 O(K_0)/O(K_0)$.

Let $W = \langle t, Q \rangle$, so that $E_{32} \cong W\alpha \in \mathcal{A}(T\alpha)$. Claim W is weakly closed in T with respect to H. For suppose $h \in H$ with $U = W^h \leqslant T$ and $U \neq W$. Then $U\alpha \in \mathcal{A}(T\alpha)$, so $U\alpha$ is conjugate under T to $\langle g\alpha, C_{W\alpha}(g) \rangle = U_1\alpha$ or $\langle g\alpha, y\alpha, C_{W\alpha}(\langle g, y \rangle) \rangle = U_2\alpha$, where y is a conjugate of g under S with $[y, K_1] \leqslant O(K_0)$. Now $U \cong W \cong Z_2 \times (Q_8 * Q_8)$. $Z(U_2) = \langle u, z \rangle$, where $u \in \mathcal{I}(Q)$ and $Z(U_1) \cong Z_4$, so $U = U_2$ and $t^H \cap \langle u, z \rangle$ is nonempty. This contradicts 10.8.

So W is weakly closed in T with respect to H. As $W\alpha$ is abelian, $N(W\alpha)$ controls fusion in $W\alpha$, so as $\langle t, z \rangle = Z(W) \trianglelefteq N(W)$, $t\alpha$ is weakly closed in $W\alpha$.

Claim $t\alpha$ is strongly closed in $T\alpha$. Assume not and let $s = t^h \in T - \langle z, t \rangle$. By 10.4, $s \in N(K_1 O(K_0))$. Replacing g by gt if necessary we may take g to be an involution. Thus $T_1 = \langle Q_1, g \rangle \cong D_{16}$. $gy \in C(t)$ and $(gy)\alpha$ is an involution in $S\alpha - W\alpha$ inducing an outer automorphism on K_1 and K_2. This determines $\langle gy, t, z\rangle$ up to cojugacy under K; in particular gy induces an automorphism of type $i(2, +)$ and as A_2 contains an involution of this type, gy is an involution. Thus $[g, y] = 1$, so setting

$$T_2 = \langle Q_2, y\rangle, \qquad [T_1K_1, T_2K_2] \leqslant O(K_0).$$

Moreover all involutions in $N(K_1O(K_0)) - W$ are fused uner KT to g, $t^{\varepsilon}gy$, $\varepsilon = 0$ or 1, or tgw, $w \in Q_2 - \langle z\rangle$, so we may take s to be one of these.

Suppose $s = t^{\varepsilon}gy$. Then s is of type $i(2, +)$ on L. On the other hand as $A_2 = A^g = \langle tz, B^g\rangle \not\leqslant L\langle t\rangle$ and $B = \langle z^{(M\cap L)}\rangle$, some $z' \in z^{L^g}$ induces an outer automorphism of L, and by 3.7 and Section 2, z' must be of type $i(2, +)$ on L. Hence $tz' \in s^L \subseteq t^G$. By 2.18.3, zz' is of type $i(2, -)$ in L^g, so $u = (zz')^g$ is of type $i(2, -)$ in L. Also $tz(zz') = tz' \in t^G$, so $tu = t^k$ for some $k \in G$. Finally as $z' \notin N(K_1)$ and z' is of type $i(2, +)$ on L, zz' is of type $i(4, -)$ by 2.18.5, so by 3.7, $zz' \notin z^G$. Thus $u \notin z^G$. Let $z_0 \in z^{(M\cap L)}$ centralize $O^2(C_G(\langle u, t\rangle))$. Then z_0 is of type $i(4, \pm 1)$ on L^k and by 3.7 it is of type $i(4, +)$. Thus

$$B = \langle A \cap O^2(C(\langle u, t\rangle)), u, z_0\rangle = A \cap L^k.$$

Hence $M_0 = \langle C_M(t), L^k \cap M\rangle$ acts on $z^G \cap B = z^{(L\cap M)}$ of order 5, and moves t, so as $C_M(t)^* \cong S_5$, $O_2(M_0^*) = C_{M_0}(B)^*$ is regular on tB and in particular $tz \in t^{C_M(z)}$. This is impossible as $A \not\leqslant L^g\langle tz\rangle$.

We have shown $t^{\varepsilon}gy \notin t^H$, so in particular $t^H \cap C_S(t) = \{t, tz\}$. But $y \in g^S \cap C(g)$ and $tyw_1 \in (tgw)^S \cap C(tgw)$ for suitable $w_1 \in Q_1 - \langle z\rangle$. Thus $s \neq g$ or tgw, so $t\alpha$ is strongly closed in $T\alpha$ as claimed.

$T\alpha \in \mathrm{Syl}_2(C_{H\alpha}(t\alpha))$, so as $t\alpha$ is strongly closed in $T\alpha$, $T \in \mathrm{Syl}_2(H)$ and $t\alpha \in Z^*(H\alpha)$ by the Z^*-theorem. Thus $KO(H) \trianglelefteq H$, and Corollary III in [3] contradicts the hypothesis that $F^*(G) \notin \mathrm{Chev}(3)$. Hence the lemma is at last established.

(10.10) *$B \trianglelefteq M$ and $t^G \cap A \subseteq tB$.*

Proof. By 10.8, $t^G \cap A \subseteq tB$, so as $tz \in t^G \cap A$ and $B = \langle z^{M\cap L}\rangle$, $B = \langle xy \colon x, y \in t^G \cap A\rangle \trianglelefteq M$.

(10.11) *Either*

(1) *$t^G \cap A = tB = t^M$ and $M^* = O_2(M^*)\,C_M(t)^*$ with $O_2(M^*) = C_M(B)^* \cong E_{16}$ regular on tB, or*

(2) *$t^G \cap A = t^M = \{t\} \cup (tz)^{(M\cap L)}$ is of order 6 and $M^* \cong A_6$ or S_6 acts naturally on B with A the permutation module, modulo its center.*

Proof. Let $\Delta = t^M$. $M \cap L$ has orbits $z^{(M\cap L)}$ *and* $u^{(M\cap L)}$ of order 5 and

10, while by 10.9, 10.3, and 10.5, $tz \in \Delta$. Thus either $\Delta = \{t\} \cup (tz)^{(M \cap L)}$ is of order 6 or $\Delta = tB$ is of order 16. In the first case as $C_M(t)^* \cong A_5$ or S_5, $M^* \cong A_6$ or S_6 and A is the permutation module, modulo its center. If $\Delta \neq t^G \cap A$ then by 10.10, $tu = t^x$ for some $x \in G$. Let $X = O^{3'}(C_M(tu))$. tu has support of order 3 in the representation of A as a permutation module, so $X^* \cong E_9$ and $B = [B, X]$. Thus $B \in \mathcal{U}(L^x)$ with $E_9 \leqslant \mathrm{Aut}_{L^x}(B)$, contradicting 10.7. Therefore (2) holds in this case.

So assume $\Delta = tB$. Then $|M^*| = 16\,|C_M(t)^*|$, so $|M^*|_3 = 3 = |C_{M \cap L}(z)^*|_3$. Thus $|z^M| \not\equiv 0 \bmod 3$, so $z^M \neq B^\#$. Hence $\Gamma = z^M = z^{(M \cap L)}$ is of order 5 and $M^\Gamma \cong A_5$ or S_5. As $B = \langle z^M \rangle$, $M_\Gamma = C_M(B)$ and $M_\Gamma^* = O_2(M^*) \cong E_{16}$, so that (1) holds.

11

In this section we continue the hypothesis and notation of Section 10. In addition assume

HYPOTHESIS 11.1. *$M^* \cong A_6$ or S_6.*

Let $S \leqslant T \in \mathrm{Syl}_2(M)$. Then $|T : S| = 2$ and $t^T = \{t, tz\}$, so $K_0 \trianglelefteq \langle K_0, T \rangle$. Notice $C_T(Q) = C_T(K_0/O(K_0))$. Let $T \leqslant R \in \mathrm{Syl}_2(N_G(Q))$, $D = C_R(Q)$, $K_D = O^2(C_G(D))$, and $K_{iD} = C_{K_iO(K_0)}(D)$.

(11.2) *Either*

(1) *$M^* \cong A_6$ and $C_T(Q) = \langle t, z \rangle$, or*

(2) *$M^* \cong S_6$ and $C_T(Q) = \langle t, r \rangle \cong D_8$, where r is an involution and r^* a transposition.*

Proof. This is a consequence of the structure of M described in 10.11.

(11.3) *$T = \langle C_T(Q), Q, a, x \rangle$, where $a \in A - Q$, $K_1^a = K_2$, x is an involution inducing an outer automorphism on $K_iO(K_0)/O(K_0)$, $i = 1$ and 2, $[a, C_T(Q)] = 1$, and if $C_T(Q) \neq \langle t, z \rangle$ then also $[x, C_T(Q)] = 1$.*

Proof. Let $a \in A - Q$, so that $K_1^a = K_2$. Then $S = \langle t, Q, a \rangle$ if $M^* \cong A_6$, while if $M^* \cong S_6$ then $S = \langle t, Q, a, x \rangle$, where x is an involution inducing an outer automorphism on K_i, $i = 1$ and 2. $T = \langle S, y \rangle$, where y is an involution fused into QA under M, y induces an outer automorphism on $(K \cap M)^* \cong A_4$, and we may choose $[a, y] = 1$. Replacing y by ya if necessary, we may assume y acts on $K_iO(K_0)$. Then as y induces an outer automorphism on $(K \cap M)^* \cong A_4$, y induces an outer automorphism on $K_iO(K_0)/O(K_0)$. Now if $C_T(Q) = \langle t, z \rangle$, set $x = y$. So take $C_T(Q) \neq \langle t, z \rangle$.

Then $T = SC_T(Q)$ and as $[t, x] = 1$ and $C_T(Q) \cong D_8$, $[C_T(Q), x] \leqslant \langle z \rangle$. Hence replacing x by tx if necessary, we may take $[x, C_T(Q)] = 1$.

(11.4) *D is dihedral or semidihedral.*

Proof. As $C_D(t) = \langle t, z \rangle$, this follows from a lemma of Suzuki.

(11.5) (1) *$\langle a, x, Q \rangle \trianglelefteq R$.*

(2) *If $M^* \cong A_6$ then $R = T$.*

(3) *$R \leqslant N(K_D)$ and $K_0 = K_D O(K_0)$.*

(4) *Either $R = TD$ or $R = TD\langle x_1 \rangle$, where $[x_1, K_{2D}] \leqslant O(K_D)$, x_1 induces an involutory outer automorphism on $K_{2D}/O(K_D)$ and $D = \langle t, t^{x_1} \rangle$.*

Proof. Assume $R \neq TD$ and let $y \in N_R(TD) - TD$. If $t^y \in t^{TD}$, then by a Frattini argument we may take $y \in C_R(t)$. As $S = C_{TD}(t) \in \mathrm{Syl}_2(C_G(t))$, this is impossible. So $t^y \notin t^{TD}$. On the other hand y acts on $C_{TD}(Q) = D$, so by 11.4, $D = \langle t, t^y \rangle$ is dihedral and as $t^y \notin t^T$ while $tz \in t^T$, $D \neq \langle t, z \rangle$. Thus, irrespective of whether $R = TD$, (2) holds and we may assume $D \neq \langle t, x \rangle$. A similar argument shows $|R : TD| \leqslant 2$.

Let $\langle t, z \rangle \leqslant E \leqslant D$ with E maximal subject to $K_0 = K_E O(K_0)$, where $K_E = O^2(C_G(E))$. $N_D(E)$ acts on K_E and centralizes Q, so $[K_E, N_D(E)] \leqslant O(K_E)$. Hence by maximality of E, $D = E$. $D \trianglelefteq R$, so $R \leqslant N(K_D)$, and (3) holds.

Let $u = a$ or x and Y the cyclic subgroup of index 2 in D. As $D \neq \langle t, z \rangle$, $T \cap D \cong D_8$ by (2) and 11.2. By 11.3, $[T \cap D, u] = 1$. Thus $[D, u] = 1$ or $[Y, u] = \langle z \rangle$ and $\langle u, z \rangle \trianglelefteq \langle D, u \rangle$. Hence if $R = TD$ then (1) and (4) hold, so we may take $R \neq TD$. Then by paragraph one, $|R : TD| = 2$ and from the structure of $\mathrm{Aut}(K_D/O(K_D))$, $R = \langle x_1, TD \rangle$, where $\langle x_1, a, x, D \rangle / \cong D_8$ and x_1 acts on $K_D/O(K_D)$ as prescribed in (4). By paragraph one, $D = \langle t, t^{x_1} \rangle$. x_1 acts on $C_{\langle a, x, D \rangle}(\Omega_2(Y)) = \langle a, x, Y \rangle$, and then on $\Omega_1(\langle a, x, Y \rangle) = \langle a, x, z \rangle$, so as $\langle a, x, Q \rangle \trianglelefteq TD$, (1) holds.

(11.6) $T\alpha \in \mathrm{Syl}_2(C_{H\alpha}(t\alpha))$.

Proof. $S \leqslant T$ with $S \in \mathrm{Syl}_2(C_G(t))$ and $t^T = \{t, tz\}$.

(11.7) *Let $W = C_T(Q)Q$. Then $W\alpha = J(T\alpha) \in \mathcal{O}(T\alpha)$ and $DQ\alpha = J(R\alpha)$.*

Proof. This is a consequence of 11.5.

(11.8) $t^H \cap DQ \subseteq D$.

Proof. By 10.10, $t^G \cap \langle t, Q \rangle = \{t, tz\}$. Also $\langle t, Q \rangle = C_{DQ}(t) = C_W(t)$ and $W\alpha = J(C_{T\alpha}(t\alpha))$ with $T\alpha \in \mathrm{Syl}_2(C_{H\alpha}(t\alpha))$ by 11.6 and 11.7. But if u is an involution in $DQ - D$ then there is $U\alpha \in \mathcal{U}(DQ) \cap C(u\alpha)$ and a K-conjugate of u in $C_U(u) - \langle u, z\rangle$. Hence $u \notin t^H$.

(11.9) $R \in \mathrm{Syl}_2(H)$.

Proof. By 11.7, $N_H(R) \leqslant N(DQ)$. By 11.5.3, $Q = DQ \cap O^2(C_G(\langle s, z\rangle))$ for each $s \in t^H \cap D$, so by 11.8, $N_H(DQ) \leqslant N(Q)$. Hence as $R \in \mathrm{Syl}_2(N_G(Q))$, $R \in \mathrm{Syl}_2(H)$.

(11.10) *$t^H \cap R \subseteq D$ so $\langle t^H \cap R\rangle$ is dihedral.*

Proof. Assume $s = t^h \in R - D$. By 11.8, $s \notin DQ$. By 10.4 and 11.8, s acts on K_{iD}. Then by 11.5 we may take $s \in xD$, $s \in x_1 D$, or $s \in x_1 u D$, $u_2 \in Q_2 - \langle z\rangle$.

Suppose $s \in C(t)$. Then $s \in xD$ by 11.5.4, so $s \in xD \cap D(t) = x\langle t, z\rangle$, and $T \cap D \cong D_8$. Then s is of type $i(2, +)$ on L. As

$$O^2(C_G(\langle s, t\rangle))/O(C_G(\langle s, t\rangle)) \cong A_4,$$

t is of type $i(2, \pm 1)$ on L and as t is not fused tu or u in G for u of type $i(2, -)$ in L by 10.11, we conclude t is of type $i(2, +)$ on L^h. Let $z' \in z^L$ centralize $O^2(C_L(s))$. By 2.18.6, z' is of type $i(4, \pm 1)$ on L^h and by 3.7 and 10.5, $z' \in L^h$ is of type $i(4, +)$. Now $t \in (tv)^{L^h}$ for some involution $v \in V - \langle z'\rangle$, where $V = \langle O_2(O^2(C_L(s))), z'\rangle$, since

$$V = C(t) \cap O_2(C_{L^h}(z')).$$

But v is of type $i(2, -)$ in L as $V \leqslant O_2(C_L(z'))$, so 10.11 contradicts $tv \in t^G$.

We have shown $t^H \cap C_R(t) = \{t, tz\}$. Hence $s^H \cap C_H(s) = \{x, zs\}$. Suppose $s \in xD$. Then s inverts elements u_i of order 4 in Q_i with $su_i \in s^{Q_i}$. But then $su_1u_2 \in s^Q \cap C(s) - \langle s, z\rangle$, a contradiction. So $s \in x_1 D$ or $x_1 u_2 D$. By 11.5, s induces an outer automorphism on $K_D\langle a, x\rangle/O(K_D)$. Suppose $s \in x_1 D$. Then ss^a is fused under K_D into $x\langle z\rangle$, so ss^a is of order 2. Hence $[s, s^a] = 1$, contradicting $s^H \cap C(s) = \{s, sz\}$. This leaves $s \in x_1 u_1 D$. $x_1^2 \in \langle Q, a, x, s\rangle \cap D = \langle z\rangle$ and $x_1 x_1^a$ is an involution as above, so $[x_1, x_1^a] = 1$. Thus as $s = x_1 u_2$ with $[x_1^a, u_2] = z$, we have $[s, s^a] = 1$, again a contradiction.

(11.11) $KO(H) \trianglelefteq H$.

Proof. By 11.10 and Corollary B4 in [9], $X = \langle t^H\rangle$ has dihedral or semidihedral Sylow 2-groups. In particular $H = XC_H(t)R$ with $[X, K] \leqslant O(X)$, so that $KO(H) \trianglelefteq H$.

Notice that 11.11 and Corollary III in [3] imply $F^*(G) \in \mathrm{Chev}(3)$, contrary to the hypothesis of this section. We have shown Hypothesis 11.1 leads to a contradiction, and hence established the following theorem:

THEOREM 11.12. *Hypothesis* 11.1 *is not satisfied, so* $t^M = tB$.

12

In this section we continue to assume the hypothesis and notation of Section 10. By 10.11 and 11.12, $O_2(M^*) \cong E_{16}$ is regular on $t^M = tB$. Let E be a Sylow 2-group of the preimage of $O_2(M^*)$. Then $A = C_E(A)$ and $E/A \cong E_{16}$ with $M^* = N_M(E)^*$. The map $eA \to [e,t]$ is an $(L \cap M)$-isomorphism of E/A and B, so E/A is the natural module for $C_M(t)^* \cong A_5$ or S_5. Next $E/B \cong E_{32}$ or $Q_8 * D_8$. In the latter case, $e^2 \in tB$ for some $e \in E$, so $e^2 \in C_A(e) = B$, a contradiction. As E/A is the natural module for $(M \cap L)^* \cong A_5$, $[E, M \cap L] = V$ is a complement to $\langle t \rangle$ in E. By [12], $V \cong (Z_4)^4$ or E_{2^8}. Summarizing:

(12.1) $V = [E, M \cap L] \cong E_{2^8}$ *or* $(Z_4)^4$ *is a complement to* $\langle t \rangle$ *in* E *with* B *and* V/B *the natural module for* $(L \cap M)^* \cong A_5$.

Set $N = N_G(V)$ and $N\beta = N/C(V)$.

(12.2) *There is a complement* X *to* A *in* $(M \cap L)\, S$ *such that either*

(1) $V \cong E_{2^8}$ *is the sum of two natural modules for* $X \cong A_5$ *or* S_5, *or*

(2) $V \cong (Z_4)^4$ *is isomorphic to* $[V_0, X]$ *as an* X-*operator group, where* $V_0 = O_2(Z_4$ *wreath* $X)$. t *inverts* V.

Proof. As is well known, there is a complement X_0 to B in $(M \cap L)\, S$, and if there is not a complement X to A in $(M \cap L)\, S$ then there is $s \in S \cap X_0$ with $s^2 = t$. Otherwise choose $s \in S \cap X - L$ with $s^2 = 1$.

If $V \cong E_{2^8}$ then as B and V/B are the natural modules for $X_0 \cap L = L_0 \cong A_5$, and as the natural module is projective, V is the sum of two natural L_0-modules. Indeed there are three irreducible L_0-submodules of V and among them only B is fixed by t, so replacing s by st if necessary, we may pick s to fix each of these submodules. In particular $s^2 \neq t$, and (1) holds.

So let $V \cong (Z_4)^4$. Let vB be a fixed point in $(V/B)^{\#}$ for S, so that $[v, t] = z$. Then S centralizes $\Phi(\langle v, B \rangle) = \langle v^2 \rangle$, so $v^2 = z$. Hence t inverts v, so as $V = \langle v^{L_0} \rangle$, t inverts V. *Let* $S \cap L_0 = \langle c, d \rangle$. For $U = B$ or V/B and $e \in \langle c, d \rangle^{\#}$, $C_U(e) = [U, e] \cong E_4$, so $C_V(c) = \langle v_1, v_1^d \rangle \cong Z_4 \times Z_4$, and we may take $v = v_1 v_1^d$ to generate $C_V(\langle c, d \rangle)$. Next s inverts or centralizes v, and replacing s by st if necessary, we may assume $[v, s] = 1$. Thus as t inverts v,

$s^2 \neq t$ and X exists. Indeed $|X: C_X(v)| = 5$, so V is the X-homorphic image of V_0. As $|[V_0, X]| = 2^8 = |V|$, (2) holds.

(12.3) $X \cong C_M(t)^* \cong A_5$.

Proof. If not, there is $s_0 \in \mathcal{J}(X) - \mathcal{J}(L)$. Next there is $s \in s_0 C_B(s_0)$ with s of type $i(4, -)$ on L. Notice $C_V(s) = C_V(s_0) \cong E_{2^6}$ or $(Z_4)^3$ by 12.2. On the other hand $L_2(9) \cong E(C_L(s)) \leqslant I \trianglelefteq C_G(s)$ with $I \in \mathcal{F}^*$ by 6.3. Hence $I \cong L_2(9)$, $L_2(81)$, or $\Omega_5(3)$. In the last two cases $I \in \mathscr{C}^*$, so $m(C(I)) = 1$ and in the first case $m(C(I)) = 2$ as $\langle t, s\rangle$ is Sylow in $C_G(\langle I, t\rangle)$. Thus $C_B(s) = [t, C_V(s)] \nleqslant C(I)$, so $t \notin C(I)$ and $I \cong L_2(81)$ or $\Omega_5(3)$ with $C_V(I) = 1$. This is impossible as $\mathrm{Aut}(I)$ does not contain a subgroup isomorphic to $C_V(s)$.

(12.4) $V \cong E_{2^8}$.

Proof. Assume $V \cong (Z_4)^4$. There is $t^h = s \in tX$. $C_V(s) \cong (Z_4)^2$ is a subgroup of $\mathrm{Aut}_G(L^h) \cong \Omega_5(3)$ by 12.3. But as B is the unique abelian subgroup of $S \cap L$ of order 16, this is impossible.

(12.5) (1) $C_G(V) = VO(N)$.

(2) $C_{N\beta}(t\beta) = C_N(t)\beta \cong Z_2 \times A_5$.

Proof. As V is transitive on the involutions in tV, $N_G(V\langle t\rangle) = VC_N(t)$. Then as $B = C_V(t) \in \mathrm{Syl}_2(C_G(V\langle t\rangle))$, the remarks follow.

(12.6) *If* $t^h \in N$ *then* $C_V(t^h) \in (B^h)^{C(t^h)}$ *and* $t^h \in t^N$.

Proof. Let $s = t^h \in N$ and $D = C_V(s)$. $m(D) \geqslant m(V)/2 = 4$. But by 12.3, $A = J(S)$, so $\langle s, D\rangle \in (A^h)^{C(s)}$. As $m(V) > m(C(t))$, $t^G \cap \langle s, D\rangle \subseteq sD$, so $D \in (B^h)^{C(s)}$. Also $V = J(SV)$ and $SV \in \mathrm{Syl}_2(M)$, so as $\langle s, D\rangle \trianglelefteq \langle s, V\rangle$, $V \in (V^h)^{C(s)}$, so $s \in t^N$.

(12.7) $(t\beta)^N \cap C(t\beta)$ *consists of the involutions in* $(t\beta)(X\beta)$.

Proof. Let $u \in \mathcal{J}(X)$. Then $tu \in t^G$ so by 12.6 $(tu)\beta \in (t\beta)^N$. On the other hand $C_V(u) = [V, u]$ so V is transitive on the involutions in Vu, so as $u \notin t^G$, $u\beta \notin (t\beta)^N$.

(12.8) *Either*

(1) $N/O(N) \cong (L \cap M)$ *wreath* Z_2, *or*

(2) $N/O(N)$ *is the split extension of* V *by* $O_4^+(4)$ *or* $Z_2/(Z_3 \times \Omega_4^+(4))$ *with* $E(N\beta) \cong \Omega_4^+(4)$ *acting naturally on* V.

Proof. By 12.5, 12.7, and the classification of groups with an involution whose centralizer is isomorphic to $Z_2 \times A_5$, $N\beta/O(N\beta) \cong O_4^\varepsilon(4)$ or $\mathrm{Aut}(U_3(4))$. As 13 does not divide the order of $L_8(2)$ the last case is out. Let $u \in \mathscr{I}(X)$. $tu \in t^N$ so $O(C_{N\beta}(s\beta)) = 1$ for $s = t$ and tu. Thus $[u, O(N(\beta))] = 1$, so $X = [X, u]$ centralizes $O(N(\beta))$. Thus $O^2(N\beta) = O(N\beta) \times Y\beta$, $Y\beta = E(N\beta) \cong \Omega_4^\varepsilon(4)$. If $\varepsilon = -1$, then as $m(V) = 16$, V is the natural module for $Y\beta \cong L_2(16)$ or $\Omega_4^-(4)$. As an element of order 3 in X is not fixed point free on V, it is the latter. But then as t induces an outer automorphism on $Y\beta$, $m(C_V(t)) = 6$, a contradiction. So $Y\beta = (Y_1)\beta \times (Y_2)\beta$, $(Y_i)\beta \cong L_2(4)$, $Y_1^t = Y_2$. Again as $m(V) = 8$, either $V_i = [Y_i, V]$ is of rank 4 or V is the natural module for $Y\beta \cong \Omega_4^+(4)$. In the former case (1) holds and in the latter (2) holds.

(12.9) *Let $S \leqslant R_0 \in \mathrm{Syl}_2(N)$. Then*

(1) $V = J(R_0)$.

(2) $R_0 \in \mathrm{Syl}_2(G)$.

Proof. Lemma 12.8 implies (1) and (1) implies (2).

(12.10) $t \notin O^2(G)$.

Proof. $t \notin O^2(N)$ and by 12.6, $t^G \cap N = t^N$, so the remark follows from 12.9 and Thompson transfer.

Let $R = R_0 \cap O^2(G)$. By 12.8 and 12.10, $R_0 = \langle t \rangle R$. Notice $O^2(G) = F^*(G)$ is simple. Let $Y = N^\infty$. For $u \in \mathscr{I}(X)$, and $s = t$ or tu, we have $[C_{O(N)}(s), u] = 1$, so $[u, O(N)] = 1$ and hence $Y/V \cong \Omega_4^+(4)$.

(12.11) *Y/V is the split extension of V by $\Omega_4^+(4)$ acting naturally on V.*

Proof. If not by 12.8, $Y = Y_1 \times Y_2$, $Y_1^t = Y_2$, $Y_i \cong L \cap M$. Let $R_i = R \cap Y_i$ and $V_i = V \cap Y_i$. Let $\langle z_i \rangle = Z(R_i)$ and $Z = \langle z_1, z_2 \rangle$, so that $Z = Z(R)$ and $z = z_1 z_2$. R_i has a subgroup P_i of index 2 with $P_i \cong Q_8 * Q_8$. Let $P = P_1 P_2$. Then $P/Z = J(R/Z)$ wth $R/P \cong E_4$, $m([P/Z, i]) \geqslant 2$ for each $i \in \mathscr{I}(R/P)$, and $m([P/Z, j]) = 4$ for some $j \in \mathscr{I}(R/P)$. Hence by Corollary 4 in [8], P/Z is strongly closed in R/Z with respect to $N(Z)$. By the Krull–Schmidt theorem, $N(P)$ permutes P_1 and P_2 while as $P_i = P_{i1} * P_{i2}$, $P_{ij} \cong Q_8$, $N(P_i)$ permutes P_{i1} and P_{i2}. Finally $N_N(P)$ is transitive on the P_{ij} and by an earlier argument $[P, O(N(Z))] = 1$. Hence the main theorem of [3] implies that either $P \lhd N(Z)$ or $P_{ij} \leqslant X_{ij} \trianglelefteq\trianglelefteq N(Z)$ with $X_{ij} \cong SL_2(q)$, $q > 3$ odd. In the latter case

$$X_{11} X_{11}^t \cap C(t) \cong SL_2(q),$$

which is impossible as $C_G(\langle t, z \rangle)$ is solvable. So $P \trianglelefteq N(Z)$.

Next $\langle z, t \rangle$ is Sylow in $C_G(\langle t, Q \rangle)$ and $D_8 \cong \langle z_1, t \rangle \leqslant C_G(\langle t, Q \rangle)$, so as

$z_1 \notin t^G$, $\langle z_1, t\rangle \in \mathrm{Syl}_2(C_G(Q))$. Thus we may take an element x of order 3 in K_1 to centralize Z by a Frattini argument. Then x acts on P and as $N(P)$ permutes P_1 and P_2, $x \in N(P_i)$. Indeed as $Q_1 = [Q, x]$ we may take $[P, x] = P_{11} \times P_{21}$.

Let $G_2 = C_G(P_1)\langle x\rangle$. Notice $C_{G_2}(z_2) = C_{G_2}(Z)$ and $SL_2(3) \cong \langle x\rangle P_{21}$ with $O(C_{G_2}(z_2))\, P_{21}\langle x\rangle \trianglelefteq\trianglelefteq C_{G_2}(z_2)$. Further $Y_2 \leqslant G_2$, $R_2 Z \in \mathrm{Syl}_2(G_2)$, and $P_2 \trianglelefteq C_{G_2}(z_2)$, so by Corollary III in [3], $E(G_2) = I \cong \Omega_5(3)$. Next let $P_2 \leqslant K_3 \trianglelefteq C_{E(G_2)}(z_2)$ with $K_3 \cong SL_2(3) * SL_2(3)$. Then $O(C_G(Z))\, K_3 \trianglelefteq\trianglelefteq C_G(Z)$, so by Corollary III in [3], $I \leqslant I_2 \in \mathscr{C}(C_G(z_1))$ with $I_2/Z(I_2) \cong \Omega_n^\varepsilon(3)$, and by the structure of R, $I_2 = I$. As $[P_1, I] = 1$, $m(C(I)) > 1$, whereas $I \in \mathscr{C}^*$ so $m(C(I)) = 1$, a contradiction.

This completes the proof of 12.11.

By 12.11, the structure of R is determined. As $L_4(4)$ satisfies the hypothesis of $F^*(G)$, R is isomorphic to a Sylow 2-group of $L_4(4)$. In particular $Z(R) = Z \cong E_4$ and R has a normal subgroup P which is the central product of two copies of a Sylow 2-group of $L_3(4)$. Then $P/Z \cong E_{2^8}$ and $R/P \cong E_4$ with $m([P/Z, i]) = 4$ for each $i \in R - P$. Hence $P/Z = J(R/Z)$ and by Corollary 4 in [8], P/Z is strongly closed in R/Z with respect to $N(Z)$. The usual argument shows $[P, O(N(Z))] = 1$. Then the main theorem of [8] and the structure of P shows

(12.12) $P \trianglelefteq N_G(Z)$.

Next an argument in the proof of 12.11 shows

(12.13) $K \leqslant C(Z)$.

(12.14) (1) *$P = U_1 U_2$ with $U_1 \cap U_2 = Z$, $U_i \cong E_{64}$.*

(2) $N_Y(U_i)/V \cong L_2(4)$.

(3) $U_i \trianglelefteq N_{F^*(G)}(P)$.

(4) $N_{F^*(G)}(Z)/O(N_G(Z))\, P \cong E_9 \times L_2(4)$.

(5) $[N_G(Z)^\infty, O(N(Z))] = 1$.

(6) *U_i/Z is the natural module for $N_G(Z)^\infty/P$.*

(7) *Every involution in R is fused into V under $C(Z)$.*

Proof. $[Z, t] = C_Z(t)$ and $[P/Z, t] = C_{P/Z}(t)$, so setting $D = N_{F^*(G)}(Z)$ and $D\gamma = D/O(D)\, P$, $C_{D_\gamma}(t\gamma) = C_D(t)\,\gamma \cong S_3$. Also $R \in \mathrm{Syl}_2(D)$ with $R\gamma \cong E_4$. It follows from the classification of groups with a Sylow 2-group isomorphic to E_4 that $D\gamma/O(D\gamma) \cong L_2(4)$. The usual argument shows D^∞ centralizes $O(D\gamma)$ and $O(D)$. Let $a \in R \cap A - P$. a inverts x of order 3 in K, so $x \in D^\infty$. $C_{Q/\langle z\rangle}(x) = 1$ and x has an equivalent action on P/QZ, so $C_{P/Z}(x) = 1$. Thus by a lemma of G. Higman, P/Z is the sum of two natural modules for D^∞/P. Hence $O^2(C_{GL(P/Z)}(D^\infty)) \cong Z_3 \times GL_2(4)$. a centralizes a Sylow 3-group E of $N_{N^\infty}(Z)$, so $[E\gamma, (D^\infty)\gamma] = 1$. Hence as $E\gamma \cong E_9$, (4) holds.

Let $Y = Y_1 Y_2$ with $V \leqslant Y_i \trianglelefteq Y$ and $Y_i/V \cong L_2(4)$. Set $V_i = \langle Z^{Y_i}\rangle$, and $U_i = \langle V_i^D \rangle$, so that V_i is the natural module for Y_i/V. $V_i = [V_i, E]$ and $[V_i, R] \leqslant Z$, so V_i is invariant under $N_D(R)$ and U_i/Z is the natural module for $(D^\infty)\gamma$. In particular D is transitive on $(U_i/Z)^\#$, so as $Z \leqslant Z(P)$, $\Phi(U_i) = 1$. Thus $U_i \cong E_{64}$, and as $V_1 \neq V_2$, $U_1 \neq U_2$, so $U_1 \cap U_2 = Z$ and $P = U_1 U_2$.

It remains to establish (2) and (7). $C_P(V_i) = V_i * P_i$, P_i isomorphic to a Sylow 2-group of $L_3(4)$. Thus P_i has two E_{16}-subgroups P_{i1} and P_{i2}, and we may take $U_i = V_i P_{i1}$. Recall V is the natural module for $\Omega_4^+(4)$ and Z is a singular point in that orthogonal space. The subspace $Z^\perp$ of V orthogonal to the point Z consists of those $v \in V$ with $[R, v] \leqslant Z$, so that $Z^\perp = P \cap V$. Next $R \cap Y_{3-i} = R_i = C_R(V_i) = V(R_i \cap P)$ and R_i has two maximal elementary abelian 2-subgroups: V and $W_i \cong E_{64}$. Hence $\{W_i, Z^\perp\} = \{U_i, V_i P_{12}\}$ and $Y_i \leqslant N(W_i)$. As $R = PV$, $Z^\perp/Z = C_{P/V}(R)$, so $U_i = W_i$ and (2) holds. Regarding P/Z as an orthogonal space over $GF(4)$, P/Z has 25 singular points. On the other hand P/Z is a sum of two natural modules for $(D^\infty)\gamma$, so there are 5 D^∞-irreducibles on P/V, which exhausts the 25 points. As $Z^\perp/Z = C_{P/Z}(R)$, each involution in P is fused into $Z^\perp$ under D. Finally each involution in $R - P$ is fused into vZ, $v \in V - Z^\perp$ under D, so (7) holds.

(12.15) *Z is a TI-set in G.*

Proof. $V = J(R)$, so $C_N(z)$ controls fusion in V with respect to $C_G(z)$. So as $Z \trianglelefteq C_N(z)$, Z is strongly closed in V with respect to $C_G(z)$. Then by 12.14.7, Z is strongly closed in R with respect to $C_G(z)$. Hence $ZO(C_G(z)) \trianglelefteq C_G(z)$ by the Z^*-theorem, while by the usual argument $[R, O(C(z))] = 1$, so $Z \trianglelefteq C_G(z)$. Then as $N_M(Z)$ is transitive on $Z^\#$, the lemma holds.

(12.16) (1) $N_G(U_i)/U_i O(N(U_i)) \cong GL_3(4)$.

(2) $U_i O(N(U_i)) = C(U_i)$.

(3) *$\mathrm{Aut}_G(U_i)$ acts naturally on U_i.*

Proof. $\mathrm{Aut}_N(U_i) \cong E_9 L_2(4)/E_{16}$ is transitive on $V_i^\#$ while

$$\mathrm{Aut}_{N(Z)}(U_i) \cong E_9 L_2(4)/E_{16}$$

and $N(Z)$ is transitive on $(U_i/Z)^\#$. Thus $N_G(U_i)$ is transitive on $U_i^\#$ and upon the 21 conjugates of the TI-set Z which partition U_i. Hence the remarks hold.

(12.17) *U_i is a weakly closed TI-set in $F^*(G)$.*

Proof. Let $D = F^*(G)$. By 12.14 and 12.15, $U_i \trianglelefteq C_D(z)$, while by 12.16 $N_G(U_i)$ is transitive on $U_i^{\#}$, so U_i is a *TI*-set in D. As $U_i \in \mathrm{Syl}_2(C_G(U_i))$ and U_i is *TI*-set in D, U_i is weakly closed in $N_G(U_i)$ with respect to D.

(12.18) $F^*(G) \cong L_4(4)$.

Proof. This follows from 12.17, the main theorem of [14], and the local structure of G.

Notice that 12.18 completes the analysis of the case $L \cong \Omega_5(3)$, and as we have already obtained contradictions when $L \in \mathscr{F}$ but $L \not\cong \Omega_5(3)$ under the assumption that $F^*(G) \notin \mathrm{Chev}(3)$, the proof of the Main Theorem is complete.

References

1. M. Aschbacher, A characterization of certain Chevalley groups and its application to component type groups, *in* "Proceedings, Conference on Finite Groups," Academic Press, New York, 1976.
2. M. Aschbacher, 2-Components in finite groups, *Comm. Algebra* **3** (1975), 901–911.
3. M. Aschbacher, A characterization of Chevalley groups over fields of odd order, *Ann. of Math.* **106** (1977), 353–468.
4. M. Aschbacher and G. Seitz, On groups with a standard component of known type, *Osaka J. Math.* **13** (1976), 439–482.
5. L. Finkelstein, Finite groups with a standard component whose centralizer has cyclic Sylow 2-groups, *Proc. Amer. Math. Soc.* **62** (1977), 237–242.
6. L. Finkelstein, Finite groups with a standard component isomorphic to M_{23}, *J. Algebra* **40** (1976), 541–555.
7. R. Foote, Finite groups with Sylow 2-subgroups of type $L_6(q)$, $q \equiv 3 \bmod 4$, preprint.
8. D. Goldschmidt, 2-Fusion in finite groups, *Ann. of Math.* **99** (1974), 70–117.
9. D. Goldschmidt, Strongly closed 2-subgroups of finite groups, *Ann. of Math.* **102** (1975), 475–489.
10. K. Gomi, Finite groups with a standard subgroup isomorphic to $PSU(4, 2)$, *Pacific J. Math.* **79** (1978), 399–462.
11. M. Harris, On balanced groups, the $B(G)$-conjecture and Chevalley groups over fields of odd order, preprint.
12. A. Prince, On 2-groups admitting A_5 or A_6 with an element of order 5 acting fixed point freely, *J. Algebra* **49** (1977), 374–386.
13. H. Suzuki, Thesis, Ohio State University.
14. F. Timmesfeld, Groups with weakly closed *TI*-subgroups, *Math. Z.* **143** (1975), 243–278.
15. J. Walter, Characterization of Chevalley groups, preprint.
16. M. Aschbacher, Finite groups with a standard subgroup isomorphic to the Tits group, preprint.